

## 递推

\$令 $f_k(n) = \sum_{i=1}^n i^k$

$$\begin{aligned} & \$ \begin{aligned} f_{k+1}(n+1) &= f_{k+1}(n) + (n+1)^{k+1} \\ &= 1 + \sum_{i=1}^n (i+1)^{k+1} \\ &= 1 + \sum_{i=1}^n \sum_{j=0}^{k+1} \binom{k+1}{j} i^j \\ &= 1 + \sum_{i=0}^{k+1} \binom{k+1}{i} f_i(n) \\ &= 1 + f_{k+1}(n) + (k+1)f_k(n) + \sum_{i=0}^{k-1} \binom{k+1}{i} f_i(n) \end{aligned} \$ $ 得到了:  $(k+1)f_k(n) = (n+1)^{k+1} - \sum_{i=0}^{k-1} \binom{k+1}{i} f_i(n)$ 

$可以写成:  $f_{k-1}(n) = \frac{(n+1)^k - 1}{k} - \frac{1}{k} \sum_{i=0}^{k-2} \binom{k-1}{i} f_i(n)$$$

## 拉格朗日插值法

对于 $\sum_{i=1}^n i^k = f[n][k]$ 具体表达是一个 $k+1$ 次多项式，我们求解 $f[0 \sim k+1][k]$ 然后进行插值即可

对于具体计算 $f[n] = \sum_{i=0}^k f[i] \frac{\prod_{j=0, j \neq i}^{k+1} (n-j)}{\prod_{j=0, j \neq i}^{k+1} (i-j)}$

$\prod_{j=1, j \neq i}^{k+1} (n-j) = \prod_{j=1}^{i-1} (n-j) \prod_{j=i+1}^{k+1} (n-j)$ 是前缀积 $\times$ 后缀积

$\prod_{j=0, j \neq i}^{k+1} (i-j) = i! (k+1-i)! (-1)^{k+1-i}$

于是，可以 $O(k)$ 预处理 $f[]$ 以及阶乘，前后缀然后 $O(k)$ 计算

这种方法对不同 $f[n]$ 计算速度一样，而针对多次询问则需要利用生成函数计算伯努利数

## 伯努利数以及生成函数

$B_0=1, B_1=-\frac{1}{2}, B_2=\frac{1}{4}, B_3=0, B_4=-\frac{1}{30}, B_5=0, B_6=\frac{1}{42}, B_7=0, B_8=-\frac{1}{30}$

\$可以利用:  $B_0=1, \sum_{i=0}^n B_i \binom{n+1}{i} = 0 \Rightarrow B_n = -\frac{1}{n+1} \sum_{i=0}^{n-1} B_i \binom{n+1}{i}$  计算\$

$\begin{aligned} & \$ \begin{aligned} \sum_{i=0}^{n-1} B_i \binom{n+1}{i} &= 0 \\ B_n &= -\frac{1}{n+1} \sum_{i=0}^{n-1} B_i \binom{n+1}{i} \\ &= -\frac{B_n}{n+1} \end{aligned} \$ \$ 考虑 C(x) = \sum_{i=0}^{\infty} \frac{B_i}{i!} x^i \\ & \text{有 } e^x C(x) = C(x) + x \\ & \text{得到关于 } \frac{C(x)}{e^x} = C(x) + x \quad (\text{加 } x \text{ 是因为 } n > 1 \text{ 导致第一项缺失, 而 } n=0 \text{ 时上述等式也是成立的}) \end{aligned}$

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