

## 递推

$$\text{\$}\text{\$}f_{k}(n)=\sum_{i=1}^n i^k \text{\$}$$

$$\begin{aligned} f_{k+1}(n+1) &= f_{k+1}(n) + (n+1)^{k+1} \\ &= 1 + \sum_{i=1}^n (i+1)^{k+1} \\ &= 1 + \sum_{j=0}^{k+1} \binom{k+1}{j} \sum_{i=1}^n i^j \\ &= 1 + \sum_{i=0}^{k+1} \binom{k+1}{i} f_i(n) \\ &= 1 + f_{k+1}(n) + (k+1)f_k(n) + \sum_{i=0}^{k-1} \binom{k+1}{i} f_i(n) \end{aligned}$$

得到了:  $(k+1)f_k(n) = (n+1)^{k+1} - 1 - \sum_{i=0}^{k-1} \binom{k+1}{i} f_i(n)$

$$\text{\$}\text{\$} \text{可以写成: } f_{k-1}(n) = \frac{(n+1)^k - 1}{k} - \frac{1}{k} \sum_{i=0}^{k-2} \binom{k}{i} f_i(n) \text{\$}$$

## 拉格朗日插值法

对于  $\sum_{i=1}^n i^k = f[n][k]$  具体表达是一个  $k+1$  次多项式，我们求解  $f[0 \sim k+1][k]$  然后进行插值即可

$$\text{\$}\text{\$} \text{对于具体计算 } f[n] = \sum_{i=0}^{k+1} f[i] \frac{\prod_{j=0, j \neq i}^{k+1} (n-j)}{\prod_{j=0, j \neq i}^{k+1} (i-j)} \text{\$}$$

$$\prod_{j=1, j \neq i}^{k+1} (n-j) = \prod_{j=1}^{i-1} (n-j) \prod_{j=i+1}^{k+1} (n-j) \text{\$}$$

是前缀积  $\times$  后缀积

$$\prod_{j=0, j \neq i}^{k+1} (i-j) = i!(k+1-i)!(-1)^{k+1-i}$$

于是，可以  $O(k)$  预处理  $f[i]$  以及阶乘，前后缀然后  $O(k)$  计算

这种方法对不同  $f[n]$  计算速度一样，而针对多次询问则需要利用生成函数计算伯努利数

## 伯努利数以及生成函数

$$\begin{aligned} B_0 &= 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6} \\ B_3 &= 0, B_4 = -\frac{1}{30}, B_5 = 0 \\ B_6 &= \frac{1}{42}, B_7 = 0, B_8 = -\frac{1}{30} \end{aligned}$$

$$\text{\$}\text{\$} \text{可以利用: } B_0 = 1, \sum_{i=0}^n B_i \binom{n+1}{i} = 0 \Rightarrow B_n = -\frac{1}{n+1} \sum_{i=0}^{n-1} B_i \binom{n+1}{i} \text{ 计算}$$


$$\begin{aligned} \sum_{i=0}^{n-1} B_i \binom{n}{i} &= 0 \\ \sum_{i=0}^n B_i \binom{n}{i} &= B_n \quad (n > 1) \\ \sum_{i=0}^n \frac{1}{i!(n-i)!} B_i &= \frac{B_n}{n!} \quad (n > 1) \end{aligned}$$

$$\text{\$}\text{\$} \text{考虑 } C(x) = \sum_{i=0}^{\infty} \frac{B_i}{i!} x^i \text{\$}$$

$$\text{\$}\text{\$} \text{有 } e^x C(x) = C(x) + x \quad (\text{加 } x \text{ 是因为 } n > 1 \text{ 导致第一项缺失, 而 } n = 0 \text{ 时上述等式也是成立的)}$$

$$\text{\$}\text{\$} \text{得到关于 } \frac{B_n}{n!} \text{ 的生成函数 } \frac{x}{e^x - 1} \text{\$}$$

From:  
<https://wiki.cvbbacm.com/> - **CVBB ACM Team**

Permanent link:  
<https://wiki.cvbbacm.com/doku.php?id=2020-2021:teams:alchemist:hardict:powersum&rev=1588991754> 

Last update: **2020/05/09 10:35**