

莫比乌斯反演技巧总结

常用狄利克雷卷积

- $\epsilon = \mu * 1$ 证明：二项式定理 $(1 - 1)^2 = 0$ 。
- $\operatorname{id} = \varphi * 1$ 证明：真分数约分。
- $\varphi = \mu * \operatorname{id}$ 证明：上面式子左右卷 μ

常用套路

经典老番

- 求 $\sum_{i=1}^n \sum_{j=1}^m \gcd(i,j)$ 先枚举 $d = \gcd(i,j)$ 再套用 $\epsilon = \mu * 1$
 $= \sum_{d=1}^n d \sum_{i=1}^{\lfloor \frac{n}{d} \rfloor} \sum_{j=1}^{\lfloor \frac{m}{d} \rfloor} [\gcd(i,j) = 1]$
 $= \sum_{d=1}^n d \sum_{i=1}^{\lfloor \frac{n}{d} \rfloor} \sum_{j=1}^{\lfloor \frac{m}{d} \rfloor} \mu(\frac{i}{d}) \mu(\frac{j}{d})$
再枚举 p
 $= \sum_{d=1}^n d \sum_{p=1}^{\lfloor \frac{m}{d} \rfloor} \sum_{i=1}^{\lfloor \frac{n}{dp} \rfloor} \sum_{j=1}^{\lfloor \frac{m}{dp} \rfloor} \mu(p) \mu(\frac{i}{dp}) \mu(\frac{j}{dp})$
 $= \sum_{d=1}^n d \sum_{p=1}^{\lfloor \frac{m}{d} \rfloor} \sum_{i=1}^{\lfloor \frac{n}{dp} \rfloor} \sum_{j=1}^{\lfloor \frac{m}{dp} \rfloor} \frac{1}{\varphi(dp)}$
设 $T = dp$ 枚举 T
 $= \sum_{T=1}^m \frac{1}{\varphi(T)} \sum_{d|T} d \mu(\frac{T}{d})$
套用 $\varphi = \mu * \operatorname{id}$
 $= \sum_{T=1}^m \frac{1}{\varphi(T)} \sum_{d|T} d \mu(\frac{T}{d})$
求出欧拉函数前缀和，直接整除分块即可。
- 上述过程中最为关键的是设 $T = dp$ 枚举 T 这一步。该操作可以概括为如下等式
 $\sum_{d=1}^n f(d) \sum_{p=1}^{\lfloor \frac{m}{d} \rfloor} \frac{1}{\varphi(dp)} = \sum_{T=1}^m h(T) \sum_{d|T} f(d) g(\frac{T}{d})$
设 $F(n) = \sum_{d|n} f(d) g(\frac{n}{d})$
则原式可化为 $\sum_{T=1}^m h(T) F(T)$
如果两个函数一个可以整除分块，另一个可以用 $O(1)/O(n \log n)/O(n)/O(n^{1/2})$ 求出前缀和，那么就可以以较低时间复杂度求出答案。

结论

$\sum_{i=1}^n \sum_{ni | \gcd(i,n)=1} \frac{1}{\varphi(ni)} = \sum_{d|n} \mu(d) \sum_{i=1}^{\lfloor \frac{n}{d} \rfloor} \frac{1}{\varphi(i)}$ 证明如下

 $\sum_{i=1}^n \sum_{ni | \gcd(i,n)=1} \frac{1}{\varphi(ni)} = \sum_{d|n} \mu(d) \sum_{i=1}^{\lfloor \frac{n}{d} \rfloor} \frac{1}{\varphi(i)}$
 $= \sum_{d|n} \mu(d) \sum_{i=1}^{\lfloor \frac{n}{d} \rfloor} \frac{1}{\varphi(\frac{i}{d})} = \sum_{d|n} \mu(d) \frac{1}{\varphi(\frac{n}{d})}$
 $= \frac{1}{\varphi(n)} \sum_{d|n} d \mu(\frac{n}{d}) = \frac{1}{\varphi(n)} \varphi(n) = 1$
由 $\varphi = \mu * \operatorname{id}$ 和 $\epsilon = \mu * 1$ 可得 $\sum_{d|n} d \mu(\frac{n}{d}) = \sum_{d|n} d \epsilon(\frac{n}{d}) = \sum_{d|n} d \cdot 1 = n$

$\sum_{n=1}^{\infty} \frac{\varphi(n)}{n} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p}\right)$

结论

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