

知识点

前言

对于一元函数的极值问题相信大家都十分熟悉，但是对于多元函数的极值问题可能就会比较陌生。大家都学过淑芬怎么可能陌生呢

对于没有限制条件的多元函数来说，只需要对函数求导即可，但是若有了限制条件，即函数的值要在一定条件下才能取到，则需要用到拉格朗日乘子法。

前置知识

- 多元函数的求导
- 多元函数极值的概念
- 隐函数存在定理

引理

设函数 $f(\vec{x})$
 $\varphi(\vec{x}) = (\varphi_1(\vec{x}), \varphi_2(\vec{x}), \dots, \varphi_m(\vec{x}))$
 在区域 $D \subset \mathbb{R}^n$ ($m < n$) 内具有各个连续偏导数，再设
 $\vec{x}_0 = (x_1^0, x_2^0, \dots, x_n^0) \in D$ 为 $f(\vec{x})$ 在约束条件

$$\begin{cases} \varphi_1(\vec{x}) = 0 \\ \varphi_2(\vec{x}) = 0 \\ \dots \\ \varphi_m(\vec{x}) = 0 \end{cases}$$
 下的极值点，并且 $\varphi'(x_0)$ 的秩为 m 则存在
 常数 $\lambda_1, \lambda_2, \dots, \lambda_m \in \mathbb{R}$ 使得在 \vec{x}_0 处成立下述
 等式
$$\begin{cases} \frac{\partial f(\vec{x}_0)}{\partial x_i} + \sum_{j=1}^m \lambda_j \frac{\partial \varphi_j(\vec{x}_0)}{\partial x_i} = 0 \quad (i=1, 2, \dots, n) \\ \varphi_j(\vec{x}_0) = 0 \quad (j=1, 2, \dots, m) \end{cases}$$

证明

由于 $\varphi'(\vec{x}_0)$ 的秩为 m 我们不妨设行列

$$\frac{\partial (\varphi_1, \varphi_2, \dots, \varphi_m)}{\partial (x_{n-m+1}, x_{n-m+2}, \dots, x_n)}$$
 在
 x_0 处不为零。因此，在 \vec{x}_0 的某个邻域内唯一确定一组具有各个连续偏导数的隐函

$$\begin{cases} x_{n-m+1} = g_1(x_1, x_2, \dots, x_{n-m}) \\ x_{n-m+2} = g_2(x_1, x_2, \dots, x_{n-m}) \\ \dots \\ x_n = g_m(x_1, x_2, \dots, x_{n-m}) \end{cases}$$
 满足

$$x_j^0 = g_j(x_1^0, x_2^0, \dots, x_{n-m}^0) \quad (j=n-m+1, n-m+2, \dots, n)$$
 且有 $\varphi_k(x_1, \dots, x_{n-m}, g_1(x_1, x_2, \dots, x_{n-m}), \dots, g_m(x_1, x_2, \dots, x_{n-m})) = 0$ 将隐函数组代入 $f(\vec{x}_0)$ 得 $f(x_1, \dots, x_{n-m}, g_1(x_1, x_2, \dots, x_{n-m}), \dots, g_m(x_1, x_2, \dots, x_{n-m}))$ 因此 \vec{x}_0 是条件极值点转化为 $(x_1^0, x_2^0, \dots, x_{n-m}^0)$ 为上述函数的通常极值点。
 令 \vec{x}_0' 则对 $i=1, 2, \dots, n-m$
 有
$$\frac{\partial f(\vec{x}_0')}{\partial x_i} + \frac{\partial f(\vec{x}_0')}{\partial x_{n-m+1}} \cdots \frac{\partial f(\vec{x}_0')}{\partial x_n} + \frac{\partial g_1(\vec{x}_0')}{\partial x_i} + \dots + \frac{\partial f(\vec{x}_0')}{\partial x_{n-m+1}} \cdots \frac{\partial f(\vec{x}_0')}{\partial x_n} \frac{\partial g_m(\vec{x}_0')}{\partial x_i} = 0$$
 令

$$\vec{g}(\vec{x}) = (g_1(\vec{x}), g_2(\vec{x}), \dots, g_m(\vec{x}))^T$$
 其中

$$\vec{x}' = (x_1, x_2, \dots, x_{n-m})$$
 将上述 $n-m$ 个等式写成向量形式，
 有
$$\left(\frac{\partial f(\vec{x}_0')}{\partial x_1}, \dots, \frac{\partial f(\vec{x}_0')}{\partial x_{n-m}} \right)$$

$$m\}}\right)+\left(\frac{\partial f(\vec{x}_0)}{\partial x_{n-m+1}}\right),\dots,\frac{\partial f(\vec{x}_0)}{\partial x_n}\right)\vec{g}(\vec{x}_0)'=0\quad \left(1\right)$$
 由于 $\vec{g}(\vec{x}_0)'=-\left(\begin{array}{c} \frac{\partial \varphi_1(\vec{x}_0)}{\partial x_{n-m+1}} & \frac{\partial \varphi_1(\vec{x}_0)}{\partial x_{n-m+2}} & \cdots & \frac{\partial \varphi_1(\vec{x}_0)}{\partial x_n} \\ \frac{\partial \varphi_2(\vec{x}_0)}{\partial x_{n-m+1}} & \frac{\partial \varphi_2(\vec{x}_0)}{\partial x_{n-m+2}} & \cdots & \frac{\partial \varphi_2(\vec{x}_0)}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \varphi_m(\vec{x}_0)}{\partial x_{n-m+1}} & \frac{\partial \varphi_m(\vec{x}_0)}{\partial x_{n-m+2}} & \cdots & \frac{\partial \varphi_m(\vec{x}_0)}{\partial x_n} \end{array}\right)^{-1} \left(\begin{array}{c} \frac{\partial \varphi_1(\vec{x}_0)}{\partial x_1} & \frac{\partial \varphi_1(\vec{x}_0)}{\partial x_2} & \cdots & \frac{\partial \varphi_1(\vec{x}_0)}{\partial x_{n-m}} \\ \frac{\partial \varphi_2(\vec{x}_0)}{\partial x_1} & \frac{\partial \varphi_2(\vec{x}_0)}{\partial x_2} & \cdots & \frac{\partial \varphi_2(\vec{x}_0)}{\partial x_{n-m}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \varphi_m(\vec{x}_0)}{\partial x_1} & \frac{\partial \varphi_m(\vec{x}_0)}{\partial x_2} & \cdots & \frac{\partial \varphi_m(\vec{x}_0)}{\partial x_{n-m}} \end{array}\right)$
 $\triangleq -A^{-1}B$
 $\quad \left(2\right)$
 注意到 $-\left(\frac{\partial f(\vec{x}_0)}{\partial x_{n-m+1}}\right),\dots,\frac{\partial f(\vec{x}_0)}{\partial x_n}\right)\cdot A^{-1}$
 是一个 m 维行向量，我们可以将其记为 $-\left(\frac{\partial f(\vec{x}_0)}{\partial x_{n-m+1}}\right),\dots,\frac{\partial f(\vec{x}_0)}{\partial x_n}\right)\cdot A^{-1}=\left(\lambda_1,\lambda_2,\dots,\lambda_m\right)\quad \left(3\right)$
 将 $\left(2\right),\left(3\right)$ 代入之前的式子 $\left(1\right)$ 得 $\left(\frac{\partial f(\vec{x}_0)}{\partial x_1}\right),\dots,\frac{\partial f(\vec{x}_0)}{\partial x_{n-m}}\right)+\left(\lambda_1,\lambda_2,\dots,\lambda_m\right)\left(\begin{array}{c} \frac{\partial \varphi_1(\vec{x}_0)}{\partial x_1} & \frac{\partial \varphi_1(\vec{x}_0)}{\partial x_2} & \cdots & \frac{\partial \varphi_1(\vec{x}_0)}{\partial x_{n-m}} \\ \frac{\partial \varphi_2(\vec{x}_0)}{\partial x_1} & \frac{\partial \varphi_2(\vec{x}_0)}{\partial x_2} & \cdots & \frac{\partial \varphi_2(\vec{x}_0)}{\partial x_{n-m}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \varphi_m(\vec{x}_0)}{\partial x_1} & \frac{\partial \varphi_m(\vec{x}_0)}{\partial x_2} & \cdots & \frac{\partial \varphi_m(\vec{x}_0)}{\partial x_{n-m}} \end{array}\right)=0\quad \left(4\right)$
 另外我们可以将 $\left(3\right)$ 改写成 $\left(\frac{\partial f(\vec{x}_0)}{\partial x_{n-m+1}}\right),\dots,\frac{\partial f(\vec{x}_0)}{\partial x_n}\right)+\left(\lambda_1,\lambda_2,\dots,\lambda_m\right)\left(\begin{array}{c} \frac{\partial \varphi_1(\vec{x}_0)}{\partial x_{n-m+1}} & \frac{\partial \varphi_1(\vec{x}_0)}{\partial x_{n-m+2}} & \cdots & \frac{\partial \varphi_1(\vec{x}_0)}{\partial x_n} \\ \frac{\partial \varphi_2(\vec{x}_0)}{\partial x_{n-m+1}} & \frac{\partial \varphi_2(\vec{x}_0)}{\partial x_{n-m+2}} & \cdots & \frac{\partial \varphi_2(\vec{x}_0)}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \varphi_m(\vec{x}_0)}{\partial x_{n-m+1}} & \frac{\partial \varphi_m(\vec{x}_0)}{\partial x_{n-m+2}} & \cdots & \frac{\partial \varphi_m(\vec{x}_0)}{\partial x_n} \end{array}\right)=0\quad \left(5\right)$
 将 $\left(4\right),\left(5\right)$ 写成分量形式再加上约束条件即可证明。

拉格朗日乘子法

构造函数 $F(x_1, \dots, x_n, \lambda_1, \dots, \lambda_m) = f(\vec{x}) + \sum_{j=1}^m \lambda_j \varphi_j(\vec{x})$ 则上述求条件极值点的必要条件形式转化为 F 的通常极值的必要条件

$$\begin{cases} \frac{\partial F(\vec{x}_0)}{\partial x_i} = 0 \quad (i=1, 2, \dots, n) \\ \frac{\partial F(\vec{x}_0)}{\partial \lambda_j} = 0 \quad (j=1, 2, \dots, m) \end{cases}$$

此即拉格朗日乘子法

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