

知识点

前言

对于一元函数的极值问题相信大家都十分熟悉，但是对于多元函数的极值问题可能就会比较陌生。大家都学过淑芬怎么可能陌生呢

对于没有限制条件的多元函数来说，只需要对函数求导即可，但是若有了限制条件，即函数的值要在一定条件下才能取到，则需要用到拉格朗日乘子法。

引理

设函数 $f(\vec{x})$

$\varphi(\vec{x}) = (\varphi_1(\vec{x}), \varphi_2(\vec{x}), \dots, \varphi_m(\vec{x}))$ 在区域 $D \subset \mathbb{R}^n$ ($m < n$) 内具有各个连续偏导数，再设

$\vec{x}_0 = (x_1^0, x_2^0, \dots, x_n^0) \in D$ 为 $f(\vec{x})$ 在约束条件

$$\begin{cases} \varphi_1(\vec{x}) = 0 \\ \varphi_2(\vec{x}) = 0 \\ \vdots \\ \varphi_m(\vec{x}) = 0 \end{cases}$$

下的极值点，并且 $\varphi'(\vec{x}_0)$ 的秩为 m 则存在常数 $\lambda_1, \lambda_2, \dots, \lambda_m \in \mathbb{R}$ 使得在 \vec{x}_0 处成立下述等式

$$\begin{cases} \frac{\partial f(\vec{x}_0)}{\partial x_i} + \sum_{j=1}^m \lambda_j \frac{\partial \varphi_j(\vec{x}_0)}{\partial x_i} = 0 \quad (i=1, 2, \dots, n) \\ \varphi_j(\vec{x}_0) = 0 \quad (j=1, 2, \dots, m) \end{cases}$$

证明

由于 $\varphi'(\vec{x}_0)$ 的秩为 m 我们不妨设行列

式 $\frac{\partial(\varphi_1, \varphi_2, \dots, \varphi_m)}{\partial(x_{n-m+1}, x_{n-m+2}, \dots, x_n)}$ 在 \vec{x}_0 处不为零。因此，在 \vec{x}_0 的某个邻域内唯一确定一组具有各个连续偏导数的隐函数

$$\begin{cases} x_{n-m+1} = g_1(x_1, x_2, \dots, x_{n-m}) \\ x_{n-m+2} = g_2(x_1, x_2, \dots, x_{n-m}) \\ \dots \\ x_n = g_m(x_1, x_2, \dots, x_{n-m}) \end{cases}$$

且 $\varphi_k(x_1, \dots, x_{n-m}, g_1(x_1, x_2, \dots, x_{n-m}), \dots, g_m(x_1, x_2, \dots, x_{n-m})) = 0$ 将隐函数组代入 $f(\vec{x}_0)$

得 $f(x_1, \dots, x_{n-m}, g_1(x_1, x_2, \dots, x_{n-m}), \dots, g_m(x_1, x_2, \dots, x_{n-m}))$ 因此 \vec{x}_0 是条件极值点转化为 $(x_1^0, x_2^0, \dots, x_{n-m}^0)$ 为上述函数的通常极值点。

令 \vec{x}_0' 则对 $i=1, 2, \dots, n-m$

$$\frac{\partial f(\vec{x}_0')}{\partial x_i} + \frac{\partial f(\vec{x}_0')}{\partial x_{n-m+1}} \dots + \frac{\partial f(\vec{x}_0')}{\partial x_n} + \frac{\partial g_1(\vec{x}_0')}{\partial x_i} + \dots + \frac{\partial g_m(\vec{x}_0')}{\partial x_i} = 0$$

$\vec{g}(\vec{x}') = (g_1(\vec{x}'), g_2(\vec{x}'), \dots, g_m(\vec{x}'))^T$ 其中

$\vec{x}' = (x_1, x_2, \dots, x_{n-m})$ 将上述 $n-m$ 个等式写成向量形式，

$$\left(\frac{\partial f(\vec{x}_0')}{\partial x_1}, \dots, \frac{\partial f(\vec{x}_0')}{\partial x_{n-m}} \right) + \left(\frac{\partial f(\vec{x}_0')}{\partial x_{n-m+1}}, \dots, \frac{\partial f(\vec{x}_0')}{\partial x_n} \right) + \vec{g}(\vec{x}_0') = 0$$

由于 $\vec{g}(\vec{x}_0') = -\left(\frac{\partial f(\vec{x}_0')}{\partial x_{n-m+1}}, \dots, \frac{\partial f(\vec{x}_0')}{\partial x_n} \right)$

$$\frac{\partial \varphi_1(\vec{x}_0')}{\partial x_{n-m+1}} \dots \frac{\partial \varphi_1(\vec{x}_0')}{\partial x_n} \dots$$

$$\frac{\partial \varphi_1(\vec{x}_0')}{\partial x_{n-m+2}} \dots \frac{\partial \varphi_1(\vec{x}_0')}{\partial x_n} \dots$$

$$\frac{\partial \varphi_2(\vec{x}_0')}{\partial x_{n-m+1}} \dots \frac{\partial \varphi_2(\vec{x}_0')}{\partial x_n} \dots$$

$$\frac{\partial \varphi_2(\vec{x}_0')}{\partial x_{n-m+1}} \dots \frac{\partial \varphi_2(\vec{x}_0')}{\partial x_n} \dots$$

$$\begin{aligned} & \frac{\partial \varphi_2(\vec{x}_0)}{\partial x_{n-m+2}} \& \cdots \& \\ & \frac{\partial \varphi_2(\vec{x}_0)}{\partial x_n} \backslash \vdots \& \vdots \& \ddots \& \vdots \backslash \\ & \frac{\partial \varphi_m(\vec{x}_0)}{\partial x_{n-m+1}} \& \\ & \frac{\partial \varphi_m(\vec{x}_0)}{\partial x_{n-m+2}} \& \cdots \& \\ & \frac{\partial \varphi_m(\vec{x}_0)}{\partial x_n} \end{array} \right)^{-1} \left(\begin{array} \{ \} \frac{\partial \varphi_1(\vec{x}_0)}{\partial x_1} \& \\ \frac{\partial \varphi_1(\vec{x}_0)}{\partial x_2} \& \cdots \& \\ \frac{\partial \varphi_1(\vec{x}_0)}{\partial x_{n-m}} \} \backslash \vdots \& \vdots \& \ddots \& \vdots \backslash \\ \frac{\partial \varphi_2(\vec{x}_0)}{\partial x_1} \& \\ \frac{\partial \varphi_2(\vec{x}_0)}{\partial x_2} \& \cdots \& \\ \frac{\partial \varphi_2(\vec{x}_0)}{\partial x_{n-m}} \} \backslash \vdots \& \vdots \& \ddots \& \vdots \backslash \\ \frac{\partial \varphi_m(\vec{x}_0)}{\partial x_1} \& \\ \frac{\partial \varphi_m(\vec{x}_0)}{\partial x_2} \& \cdots \& \\ \frac{\partial \varphi_m(\vec{x}_0)}{\partial x_{n-m}} \} \end{array} \right) \triangleq -A^{-1}B \quad \left(\frac{\partial f(\vec{x}_0)}{\partial x_{n-m+1}} \}, \dots, \frac{\partial f(\vec{x}_0)}{\partial x_n} \right) \cdot A^{-1}$$
 是一个 m 维行向量，我们可以将其记为 $\left(\frac{\partial f(\vec{x}_0)}{\partial x_{n-m+1}} \}, \dots, \frac{\partial f(\vec{x}_0)}{\partial x_n} \right) \cdot A^{-1} = \left(\lambda_1, \lambda_2, \dots, \lambda_m \right) \quad \left(\frac{\partial f(\vec{x}_0)}{\partial x_1} \}, \dots, \frac{\partial f(\vec{x}_0)}{\partial x_{n-m}} \} \right) + \left(\lambda_1, \lambda_2, \dots, \lambda_m \right) \left(\frac{\partial \varphi_1(\vec{x}_0)}{\partial x_1} \& \frac{\partial \varphi_1(\vec{x}_0)}{\partial x_2} \& \cdots \& \frac{\partial \varphi_1(\vec{x}_0)}{\partial x_{n-m}} \} \backslash \frac{\partial \varphi_2(\vec{x}_0)}{\partial x_1} \& \frac{\partial \varphi_2(\vec{x}_0)}{\partial x_2} \& \cdots \& \frac{\partial \varphi_2(\vec{x}_0)}{\partial x_{n-m}} \} \backslash \frac{\partial \varphi_m(\vec{x}_0)}{\partial x_1} \& \frac{\partial \varphi_m(\vec{x}_0)}{\partial x_2} \& \cdots \& \frac{\partial \varphi_m(\vec{x}_0)}{\partial x_{n-m}} \} \right) = 0$
 另外我们可以将 $\left(\frac{\partial f(\vec{x}_0)}{\partial x_{n-m+1}} \}, \dots, \frac{\partial f(\vec{x}_0)}{\partial x_n} \right) + \left(\lambda_1, \lambda_2, \dots, \lambda_m \right) \left(\frac{\partial \varphi_1(\vec{x}_0)}{\partial x_{n-m+1}} \} \& \frac{\partial \varphi_1(\vec{x}_0)}{\partial x_{n-m+2}} \} \& \cdots \& \frac{\partial \varphi_1(\vec{x}_0)}{\partial x_n} \} \backslash \frac{\partial \varphi_2(\vec{x}_0)}{\partial x_{n-m+1}} \} \& \frac{\partial \varphi_2(\vec{x}_0)}{\partial x_{n-m+2}} \} \& \cdots \& \frac{\partial \varphi_2(\vec{x}_0)}{\partial x_n} \} \backslash \frac{\partial \varphi_m(\vec{x}_0)}{\partial x_{n-m+1}} \} \& \frac{\partial \varphi_m(\vec{x}_0)}{\partial x_{n-m+2}} \} \& \cdots \& \frac{\partial \varphi_m(\vec{x}_0)}{\partial x_n} \} \right) = 0$
 写成分量形式再加上约束条件即可证明。

拉格朗日乘子法

构造函数 $F(x_1, \dots, x_n, \lambda_1, \dots, \lambda_m) = f(\vec{x}) + \sum_{j=1}^m \lambda_j \varphi_j(\vec{x})$ 则上述求条件极值点的必要条件形式转化为 F 的通常极值的必要条件

$$\begin{cases} \frac{\partial F(\vec{x}_0)}{\partial x_i} = 0 \quad (i=1, 2, \dots, n) \\ \frac{\partial F(\vec{x}_0)}{\partial \lambda_j} = 0 \quad (j=1, 2, \dots, m) \end{cases}$$
 此即拉格朗日乘子法

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