# 知识点

## 前言

对于一元函数的极值问题相信大家都十分熟悉，但是对于多元函数的极值问题可能就会比较陌生。~~大家都学过淑芬怎么可能陌生呢~~

对于没有限制条件的多元函数来说，只需要对函数求导即可，但是若有了限制条件，即函数的值要在一定条件下才能取到，则需要用到拉格朗日乘子法。

## 引理

设函数 $f( \vec{x})$ ，${\vec{\varphi}}(\vec{x})=({\varphi}_1(\vec{x}),{\varphi}_2(\vec{x}),\cdots,{\varphi}_m(\vec{x}))$ 在区域 $D\subset \mathbb{R}^n (m<n)$ 内具有各个连续偏导数，再设 ${\vec{x_0}}=({x_1}^0,{x_2}^0,\cdots,{x_n}^0)\in D$ 为$f(\vec{x})$ 在约束条件 $$\begin{cases}{\varphi}_1(\vec{x})=0 \\{\varphi}_2(\vec{x})=0 \\ \vdots\\{\varphi}_m(\vec{x})=0\end{cases}$$下的极值点，并且 ${\varphi}'(x_0)$ 的秩为 $m$ ，则存在常数 ${\lambda}_1,{\lambda}_2,\cdots,{\lambda}_3{\in}\mathbb{R}$ ，使得在 $\vec{x_0}$ 处成立下述等式：$$\begin{cases}{\frac{\partial{f(\vec{x_0})}}{\partial{x_i}}}+\sum_{j=1}^m {\lambda}_j \frac{\partial{\varphi}_j(\vec{x_0})}{\partial{x_i}}=0\quad (i=1,2,\cdots,n) \\ \\ {\varphi}_j(\vec{x_0})=0\qquad\qquad\qquad\qquad (j=1,2,\cdots,m)\end{cases}$$

## 证明

由于 ${\varphi'(\vec{x_0})}$ 的秩为 $m$ ，我们不妨设行列式$$\frac{\partial(\varphi_1,\varphi_2,...,\varphi_m)}{\partial(x_{n-m+1},x_{n-m+2},...,x_n)}$$在 $x_0$ 处不为零。 因此，在 $\vec{x_0}$ 的某个邻域内唯一确定一组具有各个连续偏导数的隐函数$$\begin{cases}x_{n-m+1}=g_1(x_1,x_2,...,x_{n-m}),\\ x_{n-m+2}=g_2(x_1,x_2,...,x_{n-m}),\\ ......\\ x_{n}=g_m(x_1,x_2,...,x_{n-m}),\\\end{cases}$$满足 ${x_j}^0=g_j({x_1}^0,{x_2}^0,...,{x_n}^0)(j=n-m+1,n-m+2,...,n)$ 且有$$\varphi_k(x_1,...,x_{n-m},g_1(x_1,x_2,...,x_{n-m}),...,g_m(x_1,x_2,...,x_{n-m}))=0$$ 将隐函数组代入 $f(\vec{x_0})$ 得$$f(x_1,...,x_{n-m},g_1(x_1,x_2,...,x_{n-m}),...,g_m(x_1,x_2,...,x_{n-m}))$$ 因此， $\vec{x_0}$ 是条件极值点转化为 $({x_1}^0,{x_2}^0,...,{x_{n-m}}^0)$ 为上述函数的通常极值点。
令 $\vec{x_0}'$ 则对 $i=1,2,...n-m$ 有$$\frac{\partial{f(\vec{x_0})}}{\partial{x_i}}+\frac{\partial{f(\vec{x_0})}}{\partial{x_{n-m+1}}}{\cdot}\frac{\partial{g_1(\vec{x_0}')}}{\partial{x_i}}+...+\frac{\partial{f(\vec{x_0})}}{\partial{x_n}}{\cdot}\frac{\partial{g_m(\vec{x_0}')}}{\partial{x_i}}=0$$ 令 $\vec{g}(\vec{x}'=(g_1(\vec{x}'),g_2(\vec{x}'),...,g_m(\vec{x}'))^T$ ，其中 $\vec{x}'=(x_1,x_2,...,x_{n-m})$ ，将上述 $n-m$ 个等式写成向量形式，有$$\left(\frac{\partial{f(\vec{x_0})}}{\partial{x_1}},...,\frac{\partial{f(\vec{x_0})}}{\partial{x_{n-m}}}\right)+\left(\frac{\partial{f(\vec{x_0})}}{\partial{x_{n-m+1}}},...,\frac{\partial{f(\vec{x_0})}}{\partial{x_n}}\right)\vec{g}(\vec{x_0}')=0\quad \left(1\right)$$ 由于$$\vec{g}(\vec{x_0}')=-\left(\begin{array}{}\frac{\partial{\varphi_1(\vec{x_0})}}{\partial{x_{n-m+1}}} & \frac{\partial{\varphi_1(\vec{x_0})}}{\partial{x_{n-m+2}}} & \cdots & \frac{\partial{\varphi_1(\vec{x_0})}}{\partial{x_n}}\\ \frac{\partial{\varphi_2(\vec{x_0})}}{\partial{x_{n-m+1}}} &

\frac{\partial{\varphi_2(\vec{x_0})}}{\partial{x_{n-m+2}}} & \cdots & \frac{\partial{\varphi_2(\vec{x_0})}}{\partial{x_n}}\\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial{\varphi_m(\vec{x_0})}}{\partial{x_{n-m+1}}} & \frac{\partial{\varphi_m(\vec{x_0})}}{\partial{x_{n-m+2}}} & \cdots & \frac{\partial{\varphi_m(\vec{x_0})}}{\partial{x_n}}\end{array}\right)^{-1} \left(\begin{array} {}\frac{\partial{\varphi_1(\vec{x_0})}}{\partial{x_1}} & \frac{\partial{\varphi_1(\vec{x_0})}}{\partial{x_2}} & \cdots & \frac{\partial{\varphi_1(\vec{x_0})}}{\partial{x_{n-m}}}\\ \frac{\partial{\varphi_2(\vec{x_0})}}{\partial{x_1}} & \frac{\partial{\varphi_2(\vec{x_0})}}{\partial{x_2}} & \cdots & \frac{\partial{\varphi_2(\vec{x_0})}}{\partial{x_{n-m}}}\\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial{\varphi_m(\vec{x_0})}}{\partial{x_1}} & \frac{\partial{\varphi_m(\vec{x_0})}}{\partial{x_2}} & \cdots & \frac{\partial{\varphi_m(\vec{x_0})}}{\partial{x_{n-m}}}\end{array}\right)\triangleq -A^{-1}B\quad \left(2\right)$$ 注意到$$-\left(\frac{\partial{f(\vec{x_0})}}{\partial{x_{n-m+1}}},...,\frac{\partial{f(\vec{x_0})}}{\partial{x_n}}\right)\cdot A^{-1}$$ 是一个 $m$ 维行向量，我们可以将其记为$$-\left(\frac{\partial{f(\vec{x_0})}}{\partial{x_{n-m+1}}},...,\frac{\partial{f(\vec{x_0})}}{\partial{x_n}}\right)\cdot A^{-1}=\left(\lambda_1,\lambda_2,\cdots,\lambda_m\right)\quad \left(3\right)$$ 将 $\left(2\right),\left(3\right)$代入之前的式子 $\left(1\right)$ 得 $$\left(\frac{\partial{f(\vec{x_0})}}{\partial{x_1}},...,\frac{\partial{f(\vec{x_0})}}{\partial{x_{n-m}}}\right)+\left(\lambda_1,\lambda_2,\cdots,\lambda_m\right)\left(\begin{array} {}\frac{\partial{\varphi_1(\vec{x_0})}}{\partial{x_1}} & \frac{\partial{\varphi_1(\vec{x_0})}}{\partial{x_2}} & \cdots & \frac{\partial{\varphi_1(\vec{x_0})}}{\partial{x_{n-m}}}\\ \frac{\partial{\varphi_2(\vec{x_0})}}{\partial{x_1}} & \frac{\partial{\varphi_2(\vec{x_0})}}{\partial{x_2}} & \cdots & \frac{\partial{\varphi_2(\vec{x_0})}}{\partial{x_{n-m}}}\\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial{\varphi_m(\vec{x_0})}}{\partial{x_1}} & \frac{\partial{\varphi_m(\vec{x_0})}}{\partial{x_2}} & \cdots & \frac{\partial{\varphi_m(\vec{x_0})}}{\partial{x_{n-m}}}\end{array}\right)=0\quad \left(4\right)$$ 另外我们可以将 $\left(3\right)$ 改写成 $$ \left(\frac{\partial{f(\vec{x_0})}}{\partial{x_{n-m+1}}},...,\frac{\partial{f(\vec{x_0})}}{\partial{x_n}}\right)+\left(\lambda_1,\lambda_2,\cdots,\lambda_m\right)\left(\begin{array} {}\frac{\partial{\varphi_1(\vec{x_0})}}{\partial{x_{n-m+1}}} & \frac{\partial{\varphi_1(\vec{x_0})}}{\partial{x_{n-m+2}}} & \cdots & \frac{\partial{\varphi_1(\vec{x_0})}}{\partial{x_n}}\\ \frac{\partial{\varphi_2(\vec{x_0})}}{\partial{x_{n-m+1}}} & \frac{\partial{\varphi_2(\vec{x_0})}}{\partial{x_{n-m+2}}} & \cdots & \frac{\partial{\varphi_2(\vec{x_0})}}{\partial{x_n}}\\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial{\varphi_m(\vec{x_0})}}{\partial{x_{n-m+1}}} & \frac{\partial{\varphi_m(\vec{x_0})}}{\partial{x_{n-m+2}}} & \cdots & \frac{\partial{\varphi_m(\vec{x_0})}}{\partial{x_n}}\end{array}\right)=0\quad \left(5\right)$$ 将 $\left(4\right),\left(5\right)$ 写成分量形式再加上约束条件即可证明。

# 拉格朗日乘子法

构造函数 $F(x_1,\cdots,x_n,\lambda_1,\cdots,\lambda_m)=f(\vec{x})+\sum_{j=1}^m \lambda_j\varphi_j(\vec{x})$ 则上述求条件极值点的必要条件形式转化为 $F$ 的通常极值的必要条件

$$\begin{cases}\frac{\partial{F(\vec{x_0})}}{\partial{x_i}}=0\quad(i=1,2,\cdots,n)\\
\frac{\partial{F(\vec{x_0})}}{\partial{\lambda_j}}=0\quad(j=1,2,\cdots,m)\end{cases}$$ 此即拉格朗日乘子法