

# 知识点

## 前言

对于一元函数的极值问题相信大家都十分熟悉，但是对于多元函数的极值问题可能就会比较陌生。大家都学过淑芬怎么可能陌生呢

对于没有限制条件的多元函数来说，只需要对函数求导即可，但是若有了限制条件，即函数的值要在一定条件下才能取到，则需要用到拉格朗日乘子法。

## 引理

设函数  $f(\vec{x})$

$\varphi(\vec{x}) = (\varphi_1(\vec{x}), \varphi_2(\vec{x}), \dots, \varphi_m(\vec{x}))$  在区域  $D \subset \mathbb{R}^n$  ( $m < n$ ) 内具有各个连续偏导数，再设

$\vec{x}_0 = (x_1^0, x_2^0, \dots, x_n^0) \in D$  为  $f(\vec{x})$  在约束条件

$$\begin{cases} \varphi_1(\vec{x}) = 0 \\ \varphi_2(\vec{x}) = 0 \\ \vdots \\ \varphi_m(\vec{x}) = 0 \end{cases}$$
 下的极值点，并且  $\varphi'(x_0)$  的秩为  $m$

则存在常数  $\lambda_1, \lambda_2, \dots, \lambda_m \in \mathbb{R}$  使得在  $\vec{x}_0$  处成立下述等式

$$\begin{cases} \frac{\partial f(\vec{x}_0)}{\partial x_i} + \sum_{j=1}^m \lambda_j \frac{\partial \varphi_j(\vec{x}_0)}{\partial x_i} = 0 \quad (i=1, 2, \dots, n) \\ \varphi_j(\vec{x}_0) = 0 \quad (j=1, 2, \dots, m) \end{cases}$$

## 证明

由于  $\varphi'(x_0)$  的秩为  $m$  我们不妨设行列

式  $\frac{\partial(\varphi_1, \varphi_2, \dots, \varphi_m)}{\partial(x_{n-m+1}, x_{n-m+2}, \dots, x_n)}$  在  $x_0$  处不为零。因此，在  $\vec{x}_0$  的某个邻域内唯一确定一组具有各个连续偏导数的隐函数  $\begin{cases} x_{n-m+1} = g_1(x_1, x_2, \dots, x_{n-m}) \\ x_{n-m+2} = g_2(x_1, x_2, \dots, x_{n-m}) \\ \vdots \\ x_n = g_m(x_1, x_2, \dots, x_{n-m}) \end{cases}$  满足  $x_j^0 = g_j(x_1^0, x_2^0, \dots, x_{n-m}^0)$  ( $j=n-m+1, n-m+2, \dots, n$ ) 且有  $\varphi_k(x_1, \dots, x_{n-m}, g_1(x_1, x_2, \dots, x_{n-m}), \dots, g_m(x_1, x_2, \dots, x_{n-m})) = 0$

将隐函数组代入  $f(\vec{x}_0)$  得  $f(x_1, \dots, x_{n-m}, g_1(x_1, x_2, \dots, x_{n-m}), \dots, g_m(x_1, x_2, \dots, x_{n-m}))$  因此  $\vec{x}_0$  是条件极值点转化为  $(x_1^0, x_2^0, \dots, x_{n-m}^0)$  为上述函数的通常极值点。

令  $\vec{x}_0'$  则对  $i=1, 2, \dots, n-m$

$$\frac{\partial f(\vec{x}_0')}{\partial x_i} + \frac{\partial f(\vec{x}_0')}{\partial x_{n-m+1}} \frac{\partial g_1(\vec{x}_0')}{\partial x_i} + \dots + \frac{\partial f(\vec{x}_0')}{\partial x_n} \frac{\partial g_m(\vec{x}_0')}{\partial x_i} = 0$$
 令

$\vec{g}(\vec{x}') = (g_1(\vec{x}'), g_2(\vec{x}'), \dots, g_m(\vec{x}'))^T$  其中

$\vec{x}' = (x_1, x_2, \dots, x_{n-m})$  将上述  $n-m$  个等式写成向量形式，

$$\left( \frac{\partial f(\vec{x}_0')}{\partial x_1}, \dots, \frac{\partial f(\vec{x}_0')}{\partial x_{n-m}} \right) + \left( \frac{\partial f(\vec{x}_0')}{\partial x_{n-m+1}}, \dots, \frac{\partial f(\vec{x}_0')}{\partial x_n} \right) \vec{g}(\vec{x}_0') = 0$$

由于  $\vec{g}(\vec{x}_0') = -\left( \begin{array}{c} \frac{\partial \varphi_1(\vec{x}_0')}{\partial x_{n-m+1}} \\ \frac{\partial \varphi_1(\vec{x}_0')}{\partial x_{n-m+2}} \\ \vdots \\ \frac{\partial \varphi_1(\vec{x}_0')}{\partial x_n} \end{array} \right)$

$$\frac{\partial \varphi_1(\vec{x}_0')}{\partial x_{n-m+1}} \frac{\partial f(\vec{x}_0')}{\partial x_1} + \frac{\partial \varphi_1(\vec{x}_0')}{\partial x_{n-m+2}} \frac{\partial f(\vec{x}_0')}{\partial x_1} + \dots + \frac{\partial \varphi_1(\vec{x}_0')}{\partial x_n} \frac{\partial f(\vec{x}_0')}{\partial x_1} + \dots + \frac{\partial \varphi_m(\vec{x}_0')}{\partial x_{n-m+1}} \frac{\partial f(\vec{x}_0')}{\partial x_1} + \dots + \frac{\partial \varphi_m(\vec{x}_0')}{\partial x_n} \frac{\partial f(\vec{x}_0')}{\partial x_1} = 0$$

$$\frac{\partial \varphi_2(\vec{x}_0)}{\partial x_{n-m+1}} \&$$

$$\frac{\partial \varphi_2(\vec{x}_0)}{\partial x_{n-m+2}} \& \cdots \&$$

$$\frac{\partial \varphi_2(\vec{x}_0)}{\partial x_n} \backslash \vdots \& \vdots \& \ddots \& \vdots \backslash \backslash$$

$$\frac{\partial \varphi_m(\vec{x}_0)}{\partial x_{n-m+1}} \&$$

$$\frac{\partial \varphi_m(\vec{x}_0)}{\partial x_{n-m+2}} \& \cdots \&$$

$$\frac{\partial \varphi_m(\vec{x}_0)}{\partial x_n} \end{array} \right)^{-1} \left( \begin{array} \right.$$

$$\left. \frac{\partial \varphi_1(\vec{x}_0)}{\partial x_1} \&$$

$$\frac{\partial \varphi_1(\vec{x}_0)}{\partial x_2} \& \cdots \&$$

$$\frac{\partial \varphi_1(\vec{x}_0)}{\partial x_{n-m}} \right) \backslash \backslash$$

$$\frac{\partial \varphi_2(\vec{x}_0)}{\partial x_1} \&$$

$$\frac{\partial \varphi_2(\vec{x}_0)}{\partial x_2} \& \cdots \&$$

$$\frac{\partial \varphi_2(\vec{x}_0)}{\partial x_{n-m}} \right) \backslash \vdots \& \vdots \& \ddots \& \vdots \backslash \backslash$$

$$\frac{\partial \varphi_m(\vec{x}_0)}{\partial x_1} \&$$

$$\frac{\partial \varphi_m(\vec{x}_0)}{\partial x_2} \& \cdots \&$$

$$\frac{\partial \varphi_m(\vec{x}_0)}{\partial x_{n-m}} \end{array} \right) \triangleq -$$

$$A^{-1} B \quad \text{注意到 } \left( \frac{\partial f(\vec{x}_0)}{\partial x_{n-m+1}}, \cdots, \frac{\partial f(\vec{x}_0)}{\partial x_n} \right) \cdot A^{-1}$$

是一个  $m$  维行向量，我们可以将其记为  $\left( \frac{\partial f(\vec{x}_0)}{\partial x_{n-m+1}}, \cdots, \frac{\partial f(\vec{x}_0)}{\partial x_n} \right) \cdot A^{-1} = \left( \lambda_1, \lambda_2, \cdots, \lambda_m \right)$  将  $\left( \frac{\partial f(\vec{x}_0)}{\partial x_{n-m+1}}, \cdots, \frac{\partial f(\vec{x}_0)}{\partial x_n} \right) \cdot A^{-1}$  代入之前的式子  $\left( \frac{\partial f(\vec{x}_0)}{\partial x_1}, \cdots, \frac{\partial f(\vec{x}_0)}{\partial x_{n-m}} \right) + \left( \lambda_1, \lambda_2, \cdots, \lambda_m \right) \left( \begin{array} \right.$

$$\left. \frac{\partial \varphi_1(\vec{x}_0)}{\partial x_1} \&$$

$$\frac{\partial \varphi_1(\vec{x}_0)}{\partial x_2} \& \cdots \&$$

$$\frac{\partial \varphi_1(\vec{x}_0)}{\partial x_{n-m}} \right) \backslash \backslash$$

$$\frac{\partial \varphi_2(\vec{x}_0)}{\partial x_1} \&$$

$$\frac{\partial \varphi_2(\vec{x}_0)}{\partial x_2} \& \cdots \&$$

$$\frac{\partial \varphi_2(\vec{x}_0)}{\partial x_{n-m}} \right) \backslash \vdots \& \vdots \& \ddots \& \vdots \backslash \backslash$$

$$\frac{\partial \varphi_m(\vec{x}_0)}{\partial x_1} \&$$

$$\frac{\partial \varphi_m(\vec{x}_0)}{\partial x_2} \& \cdots \&$$

$$\frac{\partial \varphi_m(\vec{x}_0)}{\partial x_{n-m}} \end{array} \right) = 0 \quad \text{另外我们可以将 } \left( \frac{\partial f(\vec{x}_0)}{\partial x_{n-m+1}}, \cdots, \frac{\partial f(\vec{x}_0)}{\partial x_n} \right) + \left( \lambda_1, \lambda_2, \cdots, \lambda_m \right) \left( \begin{array} \right.$$

$$\left. \frac{\partial \varphi_1(\vec{x}_0)}{\partial x_{n-m+1}} \&$$

$$\frac{\partial \varphi_1(\vec{x}_0)}{\partial x_{n-m+2}} \& \cdots \&$$

$$\frac{\partial \varphi_1(\vec{x}_0)}{\partial x_n} \right) \backslash \backslash$$

$$\frac{\partial \varphi_2(\vec{x}_0)}{\partial x_{n-m+1}} \&$$

$$\frac{\partial \varphi_2(\vec{x}_0)}{\partial x_{n-m+2}} \& \cdots \&$$

$$\frac{\partial \varphi_2(\vec{x}_0)}{\partial x_n} \right) \backslash \vdots \& \vdots \& \ddots \& \vdots \backslash \backslash$$

$$\frac{\partial \varphi_m(\vec{x}_0)}{\partial x_{n-m+1}} \&$$

$$\frac{\partial \varphi_m(\vec{x}_0)}{\partial x_{n-m+2}} \& \cdots \&$$

$$\frac{\partial \varphi_m(\vec{x}_0)}{\partial x_n} \end{array} \right) = 0$$

写成分量形式再加上约束条件即可证明。

## 拉格朗日乘子法

构造函数  $F(x_1, \cdots, x_n, \lambda_1, \cdots, \lambda_m) = f(\vec{x}) + \sum_{j=1}^m$

$\lambda_j \varphi_j(\vec{x})$  则上述求条件极值点的必要条件形式转化为  $F$  的通常极值的必要条件

$$\begin{cases} \frac{\partial F(\vec{x}_0)}{\partial x_i} = 0 \quad (i=1, 2, \dots, n) \\ \frac{\partial F(\vec{x}_0)}{\partial \lambda_j} = 0 \quad (j=1, 2, \dots, m) \end{cases}$$

此即拉格朗日乘子法

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