

知识点

前言

对于一元函数的极值问题相信大家都十分熟悉，但是对于多元函数的极值问题可能就会比较陌生。大家都学过淑芬怎么可能陌生呢

对于没有限制条件的多元函数来说，只需要对函数求导即可，但是若有了限制条件，即函数的值要在一定条件下才能取到，则需要用到拉格朗日乘子法。

引理

设函数 $f(\vec{x})$

$(\vec{\varphi}(\vec{x})=(\varphi_1(\vec{x}),\varphi_2(\vec{x}),\dots,\varphi_m(\vec{x})))$ 在区域 $D\subset\mathbb{R}^n (m<n)$ 内具有各个连续偏导数，再设

$(\vec{x}_0)=(x_1^0,x_2^0,\dots,x_n^0)\in D$ 为 $f(\vec{x})$ 在约束条件

$$\begin{cases} \varphi_1(\vec{x})=0 \\ \varphi_2(\vec{x})=0 \\ \vdots \\ \varphi_m(\vec{x})=0 \end{cases}$$

下的极值点，并且 $\varphi'(x_0)$ 的秩为 m 则存在常数 $(\lambda_1,\lambda_2,\dots,\lambda_3\in\mathbb{R})$ 使得在 (x_0) 处成立下述等式

$$\begin{cases} \frac{\partial f(\vec{x}_0)}{\partial x_i}+\sum_{j=1}^m \lambda_j \frac{\partial \varphi_j(\vec{x}_0)}{\partial x_i}=0 \quad (i=1,2,\dots,n) \\ \varphi_j(\vec{x}_0)=0 \quad (j=1,2,\dots,m) \end{cases}$$

证明

由于 $\varphi'(x_0)$ 的秩为 m 我们不妨设行列

式 $\frac{\partial(\varphi_1,\varphi_2,\dots,\varphi_m)}{\partial(x_{n-m+1},x_{n-m+2},\dots,x_n)}$ 在 (x_0) 处不为零。因此，在 (x_0) 的某个邻域内唯一确定一组具有各个连续偏导数的隐函数 $\begin{cases} x_{n-m+1}=g_1(x_1,x_2,\dots,x_{n-m}) \\ x_{n-m+2}=g_2(x_1,x_2,\dots,x_{n-m}) \\ \vdots \\ x_n=g_m(x_1,x_2,\dots,x_{n-m}) \end{cases}$ 满足 $x_j^0=g_j(x_1^0,x_2^0,\dots,x_{n-m}^0) (j=n-m+1,n-m+2,\dots,n)$ 且有 $\varphi_k(x_1,\dots,x_{n-m},g_1(x_1,x_2,\dots,x_{n-m}),\dots,g_m(x_1,x_2,\dots,x_{n-m}))=0$ 将隐函数组代入 $f(\vec{x}_0)$ 得 $f(x_1,\dots,x_{n-m},g_1(x_1,x_2,\dots,x_{n-m}),\dots,g_m(x_1,x_2,\dots,x_{n-m}))$ 因此 (x_0) 是条件极值点转化为 $(x_1^0,x_2^0,\dots,x_{n-m}^0)$ 为上述函数的通常极值点。

令 $(x_0)'$ 则对 $i=1,2,\dots,n-m$

$$\frac{\partial f(\vec{x}_0)}{\partial x_i}+\frac{\partial f(\vec{x}_0)}{\partial x_{n-m+1}}\cdots\frac{\partial f(\vec{x}_0)}{\partial x_n}+\frac{\partial f(\vec{x}_0)}{\partial x_i}+\cdots+\frac{\partial f(\vec{x}_0)}{\partial x_n}=\frac{\partial f(\vec{x}_0)}{\partial x_i}=0$$

令 $(g(\vec{x})=(g_1(\vec{x}),g_2(\vec{x}),\dots,g_m(\vec{x})))^T$ 其中

$(\vec{x})=(x_1,x_2,\dots,x_{n-m})$ 将上述 $n-m$ 个等式写成向量形式，

$$\left(\frac{\partial f(\vec{x}_0)}{\partial x_1},\dots,\frac{\partial f(\vec{x}_0)}{\partial x_{n-m}}\right)+\left(\frac{\partial f(\vec{x}_0)}{\partial x_{n-m+1}},\dots,\frac{\partial f(\vec{x}_0)}{\partial x_n}\right)\vec{g}(\vec{x}_0)'=0$$

由于 $(g(\vec{x}_0))'=-\left(\begin{array}{c} \frac{\partial \varphi_1(\vec{x}_0)}{\partial x_{n-m+1}} \\ \frac{\partial \varphi_1(\vec{x}_0)}{\partial x_{n-m+2}} \\ \vdots \\ \frac{\partial \varphi_1(\vec{x}_0)}{\partial x_n} \end{array}\right)$

$$\frac{\partial \varphi_2(\vec{x}_0)}{\partial x_{n-m+1}} \&$$

$$\frac{\partial \varphi_2(\vec{x}_0)}{\partial x_{n-m+2}} \& \cdots \&$$

$$\frac{\partial \varphi_2(\vec{x}_0)}{\partial x_n} \backslash \vdots \& \vdots \& \ddots \& \vdots \backslash$$

$$\frac{\partial \varphi_m(\vec{x}_0)}{\partial x_{n-m+1}} \&$$

$$\frac{\partial \varphi_m(\vec{x}_0)}{\partial x_{n-m+2}} \& \cdots \&$$

$$\frac{\partial \varphi_m(\vec{x}_0)}{\partial x_n} \end{array} \right)^{-1} \left(\begin{array} \right.$$

$$\left. \frac{\partial \varphi_1(\vec{x}_0)}{\partial x_1} \&$$

$$\frac{\partial \varphi_1(\vec{x}_0)}{\partial x_2} \& \cdots \&$$

$$\frac{\partial \varphi_1(\vec{x}_0)}{\partial x_{n-m}} \right) \backslash$$

$$\frac{\partial \varphi_2(\vec{x}_0)}{\partial x_1} \&$$

$$\frac{\partial \varphi_2(\vec{x}_0)}{\partial x_2} \& \cdots \&$$

$$\frac{\partial \varphi_2(\vec{x}_0)}{\partial x_{n-m}} \right) \backslash \vdots \& \vdots \& \ddots \& \vdots \backslash$$

$$\frac{\partial \varphi_m(\vec{x}_0)}{\partial x_1} \&$$

$$\frac{\partial \varphi_m(\vec{x}_0)}{\partial x_2} \& \cdots \&$$

$$\frac{\partial \varphi_m(\vec{x}_0)}{\partial x_{n-m}} \end{array} \right) \triangleq -$$

$A^{-1} B \quad \text{\tag{2}} \quad \text{\$ \$ 注意到 \$ \$ -} \left(\frac{\partial f(\vec{x}_0)}{\partial x_{n-m+1}}, \cdots, \frac{\partial f(\vec{x}_0)}{\partial x_n} \right) \cdot A^{-1} \text{\$ \$ 是一个 } m \text{\$ 维行向量, 我们可以将其记为 } \left(\frac{\partial f(\vec{x}_0)}{\partial x_{n-m+1}}, \cdots, \frac{\partial f(\vec{x}_0)}{\partial x_n} \right) \cdot A^{-1} = \left(\lambda_1, \lambda_2, \cdots, \lambda_m \right) \quad \text{\tag{3}} \text{\$ \$ 将 } \left(2 \right), \left(3 \right) \text{\$ 代入之前的式子 } \left(1 \right) \text{\$ 得}$

$$\left(\frac{\partial f(\vec{x}_0)}{\partial x_1}, \cdots, \frac{\partial f(\vec{x}_0)}{\partial x_{n-m}} \right) + \left(\lambda_1, \lambda_2, \cdots, \lambda_m \right) \left(\begin{array} \right.$$

$$\left. \frac{\partial \varphi_1(\vec{x}_0)}{\partial x_1} \&$$

$$\frac{\partial \varphi_1(\vec{x}_0)}{\partial x_2} \& \cdots \&$$

$$\frac{\partial \varphi_1(\vec{x}_0)}{\partial x_{n-m}} \right) \backslash$$

$$\frac{\partial \varphi_2(\vec{x}_0)}{\partial x_1} \&$$

$$\frac{\partial \varphi_2(\vec{x}_0)}{\partial x_2} \& \cdots \&$$

$$\frac{\partial \varphi_2(\vec{x}_0)}{\partial x_{n-m}} \right) \backslash \vdots \& \vdots \& \ddots \& \vdots \backslash$$

$$\frac{\partial \varphi_m(\vec{x}_0)}{\partial x_1} \&$$

$$\frac{\partial \varphi_m(\vec{x}_0)}{\partial x_2} \& \cdots \&$$

$$\frac{\partial \varphi_m(\vec{x}_0)}{\partial x_{n-m}} \end{array} \right) = 0 \quad \text{\tag{4}} \text{\$ \$ 另外$$

我们可以将 (3) 改写成 $\left(\frac{\partial f(\vec{x}_0)}{\partial x_{n-m+1}}, \cdots, \frac{\partial f(\vec{x}_0)}{\partial x_n} \right) + \left(\lambda_1, \lambda_2, \cdots, \lambda_m \right) \left(\begin{array} \right.$

$$\left. \frac{\partial \varphi_1(\vec{x}_0)}{\partial x_{n-m+1}} \&$$

$$\frac{\partial \varphi_1(\vec{x}_0)}{\partial x_{n-m+2}} \& \cdots \&$$

$$\frac{\partial \varphi_1(\vec{x}_0)}{\partial x_n} \right) \backslash$$

$$\frac{\partial \varphi_2(\vec{x}_0)}{\partial x_{n-m+1}} \&$$

$$\frac{\partial \varphi_2(\vec{x}_0)}{\partial x_{n-m+2}} \& \cdots \&$$

$$\frac{\partial \varphi_2(\vec{x}_0)}{\partial x_n} \right) \backslash \vdots \& \vdots \& \ddots \& \vdots \backslash$$

$$\frac{\partial \varphi_m(\vec{x}_0)}{\partial x_{n-m+1}} \&$$

$$\frac{\partial \varphi_m(\vec{x}_0)}{\partial x_{n-m+2}} \& \cdots \&$$

$$\frac{\partial \varphi_m(\vec{x}_0)}{\partial x_n} \end{array} \right) = 0 \quad \text{\tag{5}} \text{\$ \$ 将 } \left(4 \right), \left(5 \right) \text{\$ 写成分量形式再加上约束条件即可证明。}$$

拉格朗日乘子法

构造函数 $F(x_1, \cdots, x_n, \lambda_1, \cdots, \lambda_m) = f(\vec{x}) + \sum_{j=1}^m$

$\lambda_j \varphi_j(\vec{x})$ 则上述求条件极值点的必要条件形式转化为 F 的通常极值的必要条件

$$\begin{cases} \frac{\partial F(\vec{x}_0)}{\partial x_i} = 0 \quad (i=1, 2, \dots, n) \\ \frac{\partial F(\vec{x}_0)}{\partial \lambda_j} = 0 \quad (j=1, 2, \dots, m) \end{cases}$$

此即拉格朗日乘子法

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