

# 知识点

## 前言

对于一元函数的极值问题相信大家都十分熟悉，但是对于多元函数的极值问题可能就会比较陌生。大家都学过淑芬怎么可能陌生呢

对于没有限制条件的多元函数来说，只需要对函数求导即可，但是若有了限制条件，即函数的值要在一定条件下才能取到，则需要用到拉格朗日乘子法。

## 引理

设函数  $f(\mathbf{x})$

$\varphi_1(\mathbf{x}), \varphi_2(\mathbf{x}), \dots, \varphi_m(\mathbf{x})$  在区域  $D \subset \mathbb{R}^n$  内具有各个连续偏导数，再设  $\varphi_1(\mathbf{x}_0) = \varphi_2(\mathbf{x}_0) = \dots = \varphi_m(\mathbf{x}_0) = 0$  为约束条件。开始  $\begin{cases} \varphi_1(\mathbf{x}) = 0 \\ \varphi_2(\mathbf{x}) = 0 \\ \vdots \\ \varphi_m(\mathbf{x}) = 0 \end{cases}$  下的极值点，并且  $\varphi_i'(\mathbf{x}_0)$  的秩为  $m$ ，则存在常数  $\lambda_1, \lambda_2, \dots, \lambda_m$  使得在  $\mathbf{x}_0$  处成立下述等式  $\begin{aligned} \frac{\partial f(\mathbf{x}_0)}{\partial x_i} + \sum_{j=1}^m \lambda_j \frac{\partial \varphi_j(\mathbf{x}_0)}{\partial x_i} &= 0 \quad (i=1, 2, \dots, n) \\ \varphi_j(\mathbf{x}_0) &= 0 \quad (j=1, 2, \dots, m) \end{aligned}$

## 证明

由于  $\varphi_i'(\vec{x}_0)$  的秩为  $m$ ，我们不妨设行列

式  $\frac{\partial (\varphi_1, \varphi_2, \dots, \varphi_m)}{\partial x_1, x_2, \dots, x_n}$  在  $\mathbf{x}_0$  处不为零。因此，在  $\vec{x}_0$  的某个邻域内唯一确定一组具有各个连续偏导数的隐函数  $\begin{cases} x_{n-m+1} = g_1(x_1, x_2, \dots, x_{n-m}), \\ x_{n-m+2} = g_2(x_1, x_2, \dots, x_{n-m}), \\ \vdots \\ x_n = g_m(x_1, x_2, \dots, x_{n-m}) \end{cases}$  满足  $x_j = g_j(x_1, x_2, \dots, x_{n-m})$  ( $j=n-m+1, n-m+2, \dots, n$ ) 且有  $\varphi_k(x_1, \dots, x_{n-m}, g_1(x_1, \dots, x_{n-m}), g_2(x_1, \dots, x_{n-m}), \dots, g_m(x_1, \dots, x_{n-m})) = 0$ 。将隐函数组代入  $f(\vec{x}_0)$  得  $f(x_1, \dots, x_{n-m}, g_1(x_1, \dots, x_{n-m}), g_2(x_1, \dots, x_{n-m}), \dots, g_m(x_1, \dots, x_{n-m}))$ 。因此  $\vec{x}_0$  是条件极值点转化为  $(x_1, x_2, \dots, x_{n-m})$  为上述函数的通常极值点。

令  $\vec{x}_0'$  则对  $i=1, 2, \dots, n-m$

有  $\frac{\partial f(\vec{x}_0)}{\partial x_i} = \frac{\partial f(\vec{x}_0)}{\partial x_{n-m+1}} + \frac{\partial f(\vec{x}_0)}{\partial x_{n-m+2}} + \dots + \frac{\partial f(\vec{x}_0)}{\partial x_n}$  令  $\vec{g}(\vec{x}) = (g_1(\vec{x}), g_2(\vec{x}), \dots, g_m(\vec{x}))$  其中  $\vec{g}'(\vec{x}) = (g_1'(\vec{x}), g_2'(\vec{x}), \dots, g_m'(\vec{x}))$  将上述  $n-m$  个等式写成向量形式，有  $\left( \frac{\partial f(\vec{x}_0)}{\partial x_1}, \frac{\partial f(\vec{x}_0)}{\partial x_2}, \dots, \frac{\partial f(\vec{x}_0)}{\partial x_{n-m}} \right)' = \vec{g}'(\vec{x}_0)$  由于  $\vec{g}'(\vec{x}_0) = -\begin{pmatrix} \frac{\partial g_1}{\partial x_1}(\vec{x}_0) & \frac{\partial g_1}{\partial x_2}(\vec{x}_0) & \dots & \frac{\partial g_1}{\partial x_{n-m}}(\vec{x}_0) \\ \frac{\partial g_2}{\partial x_1}(\vec{x}_0) & \frac{\partial g_2}{\partial x_2}(\vec{x}_0) & \dots & \frac{\partial g_2}{\partial x_{n-m}}(\vec{x}_0) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g_m}{\partial x_1}(\vec{x}_0) & \frac{\partial g_m}{\partial x_2}(\vec{x}_0) & \dots & \frac{\partial g_m}{\partial x_{n-m}}(\vec{x}_0) \end{pmatrix}$

$$\begin{aligned}
 & \frac{\partial}{\partial x_1} (\varphi_1(\vec{x}_0)) \cdots \frac{\partial}{\partial x_n} (\varphi_1(\vec{x}_0)) \\
 & \frac{\partial}{\partial x_1} (\varphi_2(\vec{x}_0)) \cdots \frac{\partial}{\partial x_{n-m+1}} (\varphi_2(\vec{x}_0)) \quad \& \\
 & \frac{\partial}{\partial x_1} (\varphi_2(\vec{x}_0)) \cdots \frac{\partial}{\partial x_{n-m+2}} (\varphi_2(\vec{x}_0)) \quad \& \cdots \& \\
 & \frac{\partial}{\partial x_1} (\varphi_2(\vec{x}_0)) \cdots \frac{\partial}{\partial x_n} (\varphi_2(\vec{x}_0)) \cdots \vdots \& \vdots \& \ddots \& \vdots \\
 & \frac{\partial}{\partial x_1} (\varphi_m(\vec{x}_0)) \cdots \frac{\partial}{\partial x_{n-m+1}} (\varphi_m(\vec{x}_0)) \quad \& \\
 & \frac{\partial}{\partial x_1} (\varphi_m(\vec{x}_0)) \cdots \frac{\partial}{\partial x_{n-m+2}} (\varphi_m(\vec{x}_0)) \quad \& \cdots \& \\
 & \frac{\partial}{\partial x_1} (\varphi_1(\vec{x}_0)) \end{array}^{\text{end}} \right)^{-1} \left. \begin{array}{l} \\
 & \frac{\partial}{\partial x_1} (\varphi_1(\vec{x}_0)) \end{array} \right. \& \\
 & \frac{\partial}{\partial x_1} (\varphi_1(\vec{x}_0)) \cdots \frac{\partial}{\partial x_2} (\varphi_1(\vec{x}_0)) \quad \& \cdots \& \\
 & \frac{\partial}{\partial x_1} (\varphi_1(\vec{x}_0)) \cdots \frac{\partial}{\partial x_{n-m}} (\varphi_1(\vec{x}_0)) \cdots \vdots \& \vdots \& \ddots \& \vdots \\
 & \frac{\partial}{\partial x_1} (\varphi_2(\vec{x}_0)) \cdots \frac{\partial}{\partial x_2} (\varphi_2(\vec{x}_0)) \quad \& \cdots \& \\
 & \frac{\partial}{\partial x_1} (\varphi_2(\vec{x}_0)) \cdots \frac{\partial}{\partial x_{n-m}} (\varphi_2(\vec{x}_0)) \cdots \vdots \& \vdots \& \ddots \& \vdots \\
 & \frac{\partial}{\partial x_1} (\varphi_m(\vec{x}_0)) \cdots \frac{\partial}{\partial x_2} (\varphi_m(\vec{x}_0)) \quad \& \cdots \& \\
 & \frac{\partial}{\partial x_1} (\varphi_m(\vec{x}_0)) \cdots \frac{\partial}{\partial x_{n-m}} (\varphi_m(\vec{x}_0)) \cdots \vdots \& \vdots \& \ddots \& \vdots \\
 & \frac{\partial}{\partial x_1} (\varphi_1(\vec{x}_0)) \cdots \frac{\partial}{\partial x_2} (\varphi_1(\vec{x}_0)) \quad \& \cdots \& \\
 & \frac{\partial}{\partial x_1} (\varphi_1(\vec{x}_0)) \cdots \frac{\partial}{\partial x_{n-m}} (\varphi_1(\vec{x}_0)) \end{array} \right| \triangleq -A^{-1}B
 \end{aligned}$$

注意到  $\left( \frac{\partial}{\partial x_1} (\varphi_1(\vec{x}_0)) \cdots \frac{\partial}{\partial x_n} (\varphi_1(\vec{x}_0)) \right)$  是一个  $m$  维行向量，我们可以将其记为  $\left( \frac{\partial}{\partial x_1} (\varphi_1(\vec{x}_0)) \cdots \frac{\partial}{\partial x_n} (\varphi_1(\vec{x}_0)) \right)$ 。将  $A^{-1} = \left( \lambda_1, \lambda_2, \dots, \lambda_m \right)$  代入之前的式子  $\left( \frac{\partial}{\partial x_1} (\varphi_1(\vec{x}_0)) \cdots \frac{\partial}{\partial x_n} (\varphi_1(\vec{x}_0)) \right)$  将  $\left( \frac{\partial}{\partial x_1} (\varphi_1(\vec{x}_0)) \cdots \frac{\partial}{\partial x_n} (\varphi_1(\vec{x}_0)) \right)$  改写成  $\left( \frac{\partial}{\partial x_1} (\varphi_1(\vec{x}_0)) \cdots \frac{\partial}{\partial x_n} (\varphi_1(\vec{x}_0)) \right) + \left( \frac{\partial}{\partial x_1} (\varphi_2(\vec{x}_0)) \cdots \frac{\partial}{\partial x_n} (\varphi_2(\vec{x}_0)) \right) + \cdots + \left( \frac{\partial}{\partial x_1} (\varphi_m(\vec{x}_0)) \cdots \frac{\partial}{\partial x_n} (\varphi_m(\vec{x}_0)) \right)$ 。另外我们可以将  $\left( \frac{\partial}{\partial x_1} (\varphi_1(\vec{x}_0)) \cdots \frac{\partial}{\partial x_n} (\varphi_1(\vec{x}_0)) \right)$  改写成  $\left( \frac{\partial}{\partial x_1} (\varphi_1(\vec{x}_0)) \cdots \frac{\partial}{\partial x_n} (\varphi_1(\vec{x}_0)) \right) + \left( \frac{\partial}{\partial x_1} (\varphi_2(\vec{x}_0)) \cdots \frac{\partial}{\partial x_n} (\varphi_2(\vec{x}_0)) \right) + \cdots + \left( \frac{\partial}{\partial x_1} (\varphi_m(\vec{x}_0)) \cdots \frac{\partial}{\partial x_n} (\varphi_m(\vec{x}_0)) \right)$ 。将  $\left( \frac{\partial}{\partial x_1} (\varphi_1(\vec{x}_0)) \cdots \frac{\partial}{\partial x_n} (\varphi_1(\vec{x}_0)) \right)$  写成分量形式再加上约束条件即可证明。

## 拉格朗日乘子法

构造函数  $F(x_1, \dots, x_n, \lambda_1, \dots, \lambda_m) = f(\vec{x}) + \sum_{j=1}^m \lambda_j \varphi_j(\vec{x})$  则上述求条件极值点的必要条件形式转化为  $F$  的通常极值的必要条件  
 $\begin{cases} \frac{\partial F(\vec{x}_0)}{\partial x_i} = 0 & \text{quad}(i=1, 2, \dots, n) \\ \frac{\partial F(\vec{x}_0)}{\partial \lambda_j} = 0 & \text{quad}(j=1, 2, \dots, m) \end{cases}$  此即拉格朗日乘子法

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