

知识点

前言

对于一元函数的极值问题相信大家都十分熟悉，但是对于多元函数的极值问题可能就会比较陌生。大家都学过淑芬怎么可能陌生呢

对于没有限制条件的多元函数来说，只需要对函数求导即可，但是若有了限制条件，即函数的值要在一定条件下才能取到，则需要用到拉格朗日乘子法。

引理

设函数 $f(\mathbf{x})$

$\varphi(\mathbf{x})=(\varphi_1(\mathbf{x}),\varphi_2(\mathbf{x}),\dots,\varphi_m(\mathbf{x}))$ 在区域 $D\subset\mathbb{R}^n$ ($m<n$) 内具有各个连续偏导数，再设

$\mathbf{x}_0=(x_1^0,x_2^0,\dots,x_n^0)\in D$ 为 $f(\mathbf{x})$ 在约束条件

$$\begin{cases} \varphi_1(\mathbf{x})=0 \\ \varphi_2(\mathbf{x})=0 \\ \vdots \\ \varphi_m(\mathbf{x})=0 \end{cases}$$
 下的极值点，并且 $\varphi'(\mathbf{x}_0)$ 的秩为 m

则存在常数 $\lambda_1,\lambda_2,\dots,\lambda_m\in\mathbb{R}$ 使得在

\mathbf{x}_0 处成立下述等

式
$$\begin{cases} \frac{\partial f(\mathbf{x}_0)}{\partial x_i}+\sum_{j=1}^m\lambda_j\frac{\partial \varphi_j(\mathbf{x}_0)}{\partial x_i}=0 \\ \varphi_j(\mathbf{x}_0)=0 \end{cases} \quad (i=1,2,\dots,n) \quad (j=1,2,\dots,m)$$

证明

由于 $\varphi'(\mathbf{x}_0)$ 的秩为 m 我们不妨设行列

式
$$\frac{\partial(\varphi_1,\varphi_2,\dots,\varphi_m)}{\partial(x_{n-m+1},x_{n-m+2},\dots,x_n)}$$
 在 \mathbf{x}_0 处不为零。因此，在 \mathbf{x}_0 的某个邻域内唯一确定一组具有各个连续偏导数的隐函数

$$\begin{cases} x_{n-m+1}=g_1(x_1,x_2,\dots,x_{n-m}) \\ x_{n-m+2}=g_2(x_1,x_2,\dots,x_{n-m}) \\ \vdots \\ x_n=g_m(x_1,x_2,\dots,x_{n-m}) \end{cases}$$
 满足 $x_j^0=g_j(x_1^0,x_2^0,\dots,x_{n-m}^0)$ ($j=n-m+1,n-m+2,\dots,n$) 且有

$$\varphi_k(x_1,\dots,x_{n-m},g_1(x_1,x_2,\dots,x_{n-m}),\dots,g_m(x_1,x_2,\dots,x_{n-m}))=0$$
 将隐函数组代入 $f(\mathbf{x}_0)$

得 $f(x_1,\dots,x_{n-m},g_1(x_1,x_2,\dots,x_{n-m}),\dots,g_m(x_1,x_2,\dots,x_{n-m}))$ 因此

\mathbf{x}_0 是条件极值点转化为 $(x_1^0,x_2^0,\dots,x_{n-m}^0)$ 为上述函数的通常极值点。

令 \mathbf{x}_0' 则对 $i=1,2,\dots,n-m$

有
$$\frac{\partial f(\mathbf{x}_0')}{\partial x_i}+\frac{\partial f(\mathbf{x}_0')}{\partial x_{n-m+1}}\cdots\frac{\partial f(\mathbf{x}_0')}{\partial x_n}+\frac{\partial g_1(\mathbf{x}_0')}{\partial x_i}+\cdots+\frac{\partial g_m(\mathbf{x}_0')}{\partial x_i}=0$$
 令

$\mathbf{g}'(\mathbf{x})=(g_1'(\mathbf{x}),g_2'(\mathbf{x}),\dots,g_m'(\mathbf{x}))^T$ 其中

$\mathbf{x}'=(x_1,x_2,\dots,x_{n-m})$ 将上述 $n-m$ 个等式写成向量形式，

有
$$\left(\frac{\partial f(\mathbf{x}_0')}{\partial x_1},\dots,\frac{\partial f(\mathbf{x}_0')}{\partial x_{n-m}}\right)+\left(\frac{\partial f(\mathbf{x}_0')}{\partial x_{n-m+1}},\dots,\frac{\partial f(\mathbf{x}_0')}{\partial x_n}\right)+\mathbf{g}'(\mathbf{x}_0')=0$$
 由于 $\mathbf{g}'(\mathbf{x}_0')=-\left(\frac{\partial f(\mathbf{x}_0')}{\partial x_{n-m+1}},\dots,\frac{\partial f(\mathbf{x}_0')}{\partial x_n}\right)$ &

$$\begin{aligned} & \frac{\partial \varphi_1(\mathbf{x}_0)}{\partial x_{n-m+2}} \& \cdots \& \\ & \frac{\partial \varphi_1(\mathbf{x}_0)}{\partial x_n} \\ & \frac{\partial \varphi_2(\mathbf{x}_0)}{\partial x_{n-m+1}} \& \\ & \frac{\partial \varphi_2(\mathbf{x}_0)}{\partial x_{n-m+2}} \& \cdots \& \\ & \frac{\partial \varphi_2(\mathbf{x}_0)}{\partial x_n} \\ & \vdots \& \vdots \& \ddots \& \vdots \\ & \frac{\partial \varphi_m(\mathbf{x}_0)}{\partial x_{n-m+1}} \& \\ & \frac{\partial \varphi_m(\mathbf{x}_0)}{\partial x_{n-m+2}} \& \cdots \& \\ & \frac{\partial \varphi_m(\mathbf{x}_0)}{\partial x_n} \end{aligned} \end{array} \right)^{-1} \left(\begin{array}{l} \frac{\partial \varphi_1(\mathbf{x}_0)}{\partial x_1} \& \\ \frac{\partial \varphi_1(\mathbf{x}_0)}{\partial x_2} \& \cdots \& \\ \frac{\partial \varphi_1(\mathbf{x}_0)}{\partial x_{n-m}} \\ \frac{\partial \varphi_2(\mathbf{x}_0)}{\partial x_1} \& \\ \frac{\partial \varphi_2(\mathbf{x}_0)}{\partial x_2} \& \cdots \& \\ \frac{\partial \varphi_2(\mathbf{x}_0)}{\partial x_{n-m}} \\ \vdots \& \vdots \& \ddots \& \vdots \\ \frac{\partial \varphi_m(\mathbf{x}_0)}{\partial x_1} \& \\ \frac{\partial \varphi_m(\mathbf{x}_0)}{\partial x_2} \& \cdots \& \\ \frac{\partial \varphi_m(\mathbf{x}_0)}{\partial x_{n-m}} \end{array} \right) \triangleq -A^{-1}B \quad \tag{2}$$

注意到 $-\left(\frac{\partial f(\mathbf{x}_0)}{\partial x_{n-m+1}}, \dots, \frac{\partial f(\mathbf{x}_0)}{\partial x_n}\right) \cdot A^{-1}$ 是一个 m 维行向量，我们可以将其记为 $-\left(\frac{\partial f(\mathbf{x}_0)}{\partial x_{n-m+1}}, \dots, \frac{\partial f(\mathbf{x}_0)}{\partial x_n}\right) \cdot A^{-1} = \left(\lambda_1, \lambda_2, \dots, \lambda_m\right)$ 将 $\left(\frac{\partial \varphi_1(\mathbf{x}_0)}{\partial x_1}, \dots, \frac{\partial \varphi_1(\mathbf{x}_0)}{\partial x_{n-m}}\right) + \left(\lambda_1, \lambda_2, \dots, \lambda_m\right) \left(\frac{\partial \varphi_1(\mathbf{x}_0)}{\partial x_1} \& \frac{\partial \varphi_1(\mathbf{x}_0)}{\partial x_2} \& \cdots \& \frac{\partial \varphi_1(\mathbf{x}_0)}{\partial x_{n-m}}\right)$ 代入之前的式子 $\left(\frac{\partial \varphi_1(\mathbf{x}_0)}{\partial x_1} \& \frac{\partial \varphi_1(\mathbf{x}_0)}{\partial x_2} \& \cdots \& \frac{\partial \varphi_1(\mathbf{x}_0)}{\partial x_{n-m}}\right)$ 得

$$\left(\frac{\partial \varphi_1(\mathbf{x}_0)}{\partial x_1} \& \frac{\partial \varphi_1(\mathbf{x}_0)}{\partial x_2} \& \cdots \& \frac{\partial \varphi_1(\mathbf{x}_0)}{\partial x_{n-m}}\right) + \left(\lambda_1, \lambda_2, \dots, \lambda_m\right) \left(\frac{\partial \varphi_1(\mathbf{x}_0)}{\partial x_1} \& \frac{\partial \varphi_1(\mathbf{x}_0)}{\partial x_2} \& \cdots \& \frac{\partial \varphi_1(\mathbf{x}_0)}{\partial x_{n-m}}\right) = 0 \quad \tag{4}$$

另外我们可以将 $\left(\frac{\partial \varphi_1(\mathbf{x}_0)}{\partial x_{n-m+1}}, \dots, \frac{\partial \varphi_1(\mathbf{x}_0)}{\partial x_n}\right) + \left(\lambda_1, \lambda_2, \dots, \lambda_m\right) \left(\frac{\partial \varphi_1(\mathbf{x}_0)}{\partial x_{n-m+1}} \& \frac{\partial \varphi_1(\mathbf{x}_0)}{\partial x_{n-m+2}} \& \cdots \& \frac{\partial \varphi_1(\mathbf{x}_0)}{\partial x_n}\right)$ 写成分量形式再加上约束条件即可证明。

拉格朗日乘子法

构造函数 $F(x_1, \dots, x_n, \lambda_1, \dots, \lambda_m) = f(\mathbf{x}) + \sum_{j=1}^m \lambda_j \varphi_j(\mathbf{x})$ 则上述求条件极值点的必要条件形式转化为 F 的通常极值的必要条件
$$\begin{cases} \frac{\partial F(\mathbf{x}_0)}{\partial x_i} = 0 \quad (i=1, 2, \dots, n) \\ \frac{\partial F(\mathbf{x}_0)}{\partial \lambda_j} = 0 \quad (j=1, 2, \dots, m) \end{cases}$$
 此即拉格朗日乘子法

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