

格式

- 1. 向量建议写成 \mathbf{x}_0

内容

- 1. 没有例题吗

知识点

前言

对于一元函数的极值问题相信大家都十分熟悉，但是对于多元函数的极值问题可能就会比较陌生。大家都学过淑芬怎么可能陌生呢

对于没有限制条件的多元函数来说，只需要对函数求导即可，但是若有了限制条件，即函数的值要在一定条件下才能取到，则需要用到拉格朗日乘子法。

引理

设函数 $f(\mathbf{x})$

$\varphi(\mathbf{x})=(\varphi_1(\mathbf{x}),\varphi_2(\mathbf{x}),\dots,\varphi_m(\mathbf{x}))$ 在区域 $D\subset\mathbb{R}^n (m<n)$ 内具有各个连续偏导数，再设

$\mathbf{x}_0=(x_1^0,x_2^0,\dots,x_n^0)\in D$ 为 $f(\mathbf{x})$ 在约束条件

$$\begin{cases} \varphi_1(\mathbf{x})=0 \\ \varphi_2(\mathbf{x})=0 \\ \vdots \\ \varphi_m(\mathbf{x})=0 \end{cases}$$

下的极值点，并且 $\varphi'(\mathbf{x}_0)$ 的秩为 m

则存在常数 $\lambda_1,\lambda_2,\dots,\lambda_m\in\mathbb{R}$ 使得在

\mathbf{x}_0 处成立下述等

$$\frac{\partial f(\mathbf{x}_0)}{\partial x_i}+\sum_{j=1}^m\lambda_j\frac{\partial \varphi_j(\mathbf{x}_0)}{\partial x_i}=0 \quad (i=1,2,\dots,n)$$

$$\varphi_j(\mathbf{x}_0)=0 \quad (j=1,2,\dots,m)$$

证明

由于 $\varphi'(\mathbf{x}_0)$ 的秩为 m 我们不妨设行列

式 $\frac{\partial(\varphi_1,\varphi_2,\dots,\varphi_m)}{\partial(x_{n-m+1},x_{n-m+2},\dots,x_n)}$ 在 \mathbf{x}_0 处不为零。因此，在 \mathbf{x}_0 的某个邻域内唯一确定一组具有各个连续偏导数的隐函数

$$\begin{cases} x_{n-m+1}=g_1(x_1,x_2,\dots,x_{n-m}) \\ x_{n-m+2}=g_2(x_1,x_2,\dots,x_{n-m}) \\ \vdots \\ x_n=g_m(x_1,x_2,\dots,x_{n-m}) \end{cases}$$

满足 $x_j^0=g_j(x_1^0,x_2^0,\dots,x_{n-m}^0) (j=n-m+1,n-m+2,\dots,n)$ 且有 $\varphi_k(x_1,\dots,x_{n-m},g_1(x_1,x_2,\dots,x_{n-m}),\dots,g_m(x_1,x_2,\dots,x_{n-m}))=0$ 将隐函数组代入 $f(\mathbf{x}_0)$

得 $f(x_1,\dots,x_{n-m},g_1(x_1,x_2,\dots,x_{n-m}),\dots,g_m(x_1,x_2,\dots,x_{n-m}))$ 因此

\mathbf{x}_0 是条件极值点转化为 $(x_1^0,x_2^0,\dots,x_{n-m}^0)$ 为上述函数的通常极值点。

令 \mathbf{x}_0' 则对 $i=1,2,\dots,n-m$

$$\frac{\partial f(\mathbf{x}_0')}{\partial x_i}+\frac{\partial f(\mathbf{x}_0')}{\partial x_{n-m+1}}\cdots\frac{\partial f(\mathbf{x}_0')}{\partial x_n}+\frac{\partial g_1(\mathbf{x}_0')}{\partial x_i}+\cdots+\frac{\partial f(\mathbf{x}_0')}{\partial x_n}\frac{\partial g_m(\mathbf{x}_0')}{\partial x_i}=0$$

$\mathbf{g}(\mathbf{x})=(g_1(\mathbf{x}),g_2(\mathbf{x}),\dots,g_m(\mathbf{x}))^T$ 其中
 $\mathbf{x}=(x_1,x_2,\dots,x_{n-m})$ 将上述 $n-m$ 个等式写成向量形式，
 有 $\left(\frac{\partial f(\mathbf{x}_0)}{\partial x_1},\dots,\frac{\partial f(\mathbf{x}_0)}{\partial x_{n-m}}\right)+\left(\frac{\partial f(\mathbf{x}_0)}{\partial x_{n-m+1}},\dots,\frac{\partial f(\mathbf{x}_0)}{\partial x_n}\right)\mathbf{g}(\mathbf{x}_0)=0$
 $\tag{1}$ 由于 $\mathbf{g}(\mathbf{x}_0)=-\left(\begin{array}{c} \frac{\partial \varphi_1(\mathbf{x}_0)}{\partial x_{n-m+1}} & \dots & \frac{\partial \varphi_1(\mathbf{x}_0)}{\partial x_n} \\ \frac{\partial \varphi_2(\mathbf{x}_0)}{\partial x_{n-m+1}} & \dots & \frac{\partial \varphi_2(\mathbf{x}_0)}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial \varphi_m(\mathbf{x}_0)}{\partial x_{n-m+1}} & \dots & \frac{\partial \varphi_m(\mathbf{x}_0)}{\partial x_n} \end{array}\right)^{-1} \left(\begin{array}{c} \frac{\partial \varphi_1(\mathbf{x}_0)}{\partial x_1} & \dots & \frac{\partial \varphi_1(\mathbf{x}_0)}{\partial x_{n-m}} \\ \frac{\partial \varphi_2(\mathbf{x}_0)}{\partial x_1} & \dots & \frac{\partial \varphi_2(\mathbf{x}_0)}{\partial x_{n-m}} \\ \vdots & \ddots & \vdots \\ \frac{\partial \varphi_m(\mathbf{x}_0)}{\partial x_1} & \dots & \frac{\partial \varphi_m(\mathbf{x}_0)}{\partial x_{n-m}} \end{array}\right)$
 $\tag{2}$ 注意到 $-\left(\frac{\partial f(\mathbf{x}_0)}{\partial x_{n-m+1}},\dots,\frac{\partial f(\mathbf{x}_0)}{\partial x_n}\right)\mathbf{g}(\mathbf{x}_0)$ 是一个 m 维
 行向量，我们可以将其记为 $-\left(\frac{\partial f(\mathbf{x}_0)}{\partial x_{n-m+1}},\dots,\frac{\partial f(\mathbf{x}_0)}{\partial x_n}\right)\mathbf{g}(\mathbf{x}_0)=\left(\lambda_1,\lambda_2,\dots,\lambda_m\right)$
 $\tag{3}$ 将 $\left(\frac{\partial f(\mathbf{x}_0)}{\partial x_{n-m+1}},\dots,\frac{\partial f(\mathbf{x}_0)}{\partial x_n}\right)$ 代入之前的式子 $\left(\frac{\partial f(\mathbf{x}_0)}{\partial x_1},\dots,\frac{\partial f(\mathbf{x}_0)}{\partial x_{n-m}}\right)+\left(\lambda_1,\lambda_2,\dots,\lambda_m\right)\left(\begin{array}{c} \frac{\partial \varphi_1(\mathbf{x}_0)}{\partial x_{n-m+1}} & \dots & \frac{\partial \varphi_1(\mathbf{x}_0)}{\partial x_n} \\ \frac{\partial \varphi_2(\mathbf{x}_0)}{\partial x_{n-m+1}} & \dots & \frac{\partial \varphi_2(\mathbf{x}_0)}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial \varphi_m(\mathbf{x}_0)}{\partial x_{n-m+1}} & \dots & \frac{\partial \varphi_m(\mathbf{x}_0)}{\partial x_n} \end{array}\right)=0$
 $\tag{4}$ 另外我们可以将 $\left(\frac{\partial f(\mathbf{x}_0)}{\partial x_{n-m+1}},\dots,\frac{\partial f(\mathbf{x}_0)}{\partial x_n}\right)+\left(\lambda_1,\lambda_2,\dots,\lambda_m\right)\left(\begin{array}{c} \frac{\partial \varphi_1(\mathbf{x}_0)}{\partial x_{n-m+1}} & \dots & \frac{\partial \varphi_1(\mathbf{x}_0)}{\partial x_n} \\ \frac{\partial \varphi_2(\mathbf{x}_0)}{\partial x_{n-m+1}} & \dots & \frac{\partial \varphi_2(\mathbf{x}_0)}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial \varphi_m(\mathbf{x}_0)}{\partial x_{n-m+1}} & \dots & \frac{\partial \varphi_m(\mathbf{x}_0)}{\partial x_n} \end{array}\right)$

$$\frac{\partial \varphi_m(\mathbf{x}_0)}{\partial x_{n-m+2}} \& \cdots \& \frac{\partial \varphi_m(\mathbf{x}_0)}{\partial x_n} \end{array} \right) = 0 \quad \text{\tag{5}}$$
 将 $(4), (5)$ 写成分量形式再加上约束条件即可证明。

拉格朗日乘子法

构造函数 $F(x_1, \dots, x_n, \lambda_1, \dots, \lambda_m) = f(\mathbf{x}) + \sum_{j=1}^m \lambda_j \varphi_j(\mathbf{x})$ 则上述求条件极值点的必要条件形式转化为 F 的通常极值的必要条件

$$\begin{cases} \frac{\partial F(\mathbf{x}_0)}{\partial x_i} = 0 \quad (i=1, 2, \dots, n) \\ \frac{\partial F(\mathbf{x}_0)}{\partial \lambda_j} = 0 \quad (j=1, 2, \dots, m) \end{cases}$$
 此即拉格朗日乘子法

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Last update: 2020/06/12 19:11