

格式

- 向量建议写成 \boldsymbol{x}_0

内容

- 没有例题吗

知识点

前言

对于一元函数的极值问题相信大家都十分熟悉，但是对于多元函数的极值问题可能就会比较陌生。大家都学过淑芬怎么可能陌生呢

对于没有限制条件的多元函数来说，只需要对函数求导即可，但是若有了限制条件，即函数的值要在一定条件下才能取到，则需要用到拉格朗日乘子法。

引理

设函数 $f(\mathbf{x})$

$\varphi_1(\mathbf{x}) = (\varphi_1(\mathbf{x}), \varphi_2(\mathbf{x}), \dots, \varphi_m(\mathbf{x}))$ 在区域 $D \subset \mathbb{R}^n$ ($m < n$) 内具有各个连续偏导数，再设 $\mathbf{x}_0 = (x_1^0, x_2^0, \dots, x_n^0) \in D$ 为 $f(\mathbf{x})$ 在约束条件 $\begin{cases} \varphi_1(\mathbf{x}) = 0 \\ \varphi_2(\mathbf{x}) = 0 \\ \vdots \\ \varphi_m(\mathbf{x}) = 0 \end{cases}$ 下的极值点，并且 $\varphi'(\mathbf{x}_0)$ 的秩为 m 则存在常数 $\lambda_1, \lambda_2, \dots, \lambda_m \in \mathbb{R}$ 使得在 \mathbf{x}_0 处成立下述等式
$$\begin{cases} \frac{\partial f(\mathbf{x}_0)}{\partial x_i} + \sum_{j=1}^m \lambda_j \frac{\partial \varphi_j(\mathbf{x}_0)}{\partial x_i} = 0 & (i=1, 2, \dots, n) \\ \vdots \\ \frac{\partial f(\mathbf{x}_0)}{\partial x_n} + \sum_{j=1}^m \lambda_j \frac{\partial \varphi_j(\mathbf{x}_0)}{\partial x_n} = 0 \end{cases}$$

证明

由于 $\varphi'(\mathbf{x}_0)$ 的秩为 m 我们不妨设行列

$\begin{vmatrix} \frac{\partial \varphi_1}{\partial x_1} & \frac{\partial \varphi_1}{\partial x_2} & \cdots & \frac{\partial \varphi_1}{\partial x_n} \\ \frac{\partial \varphi_2}{\partial x_1} & \frac{\partial \varphi_2}{\partial x_2} & \cdots & \frac{\partial \varphi_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \varphi_m}{\partial x_1} & \frac{\partial \varphi_m}{\partial x_2} & \cdots & \frac{\partial \varphi_m}{\partial x_n} \end{vmatrix} \neq 0$ 在 \mathbf{x}_0 处不为零。因此，在 \mathbf{x}_0 的某个邻域内唯一确定一组具有各个连续偏导数的隐函数 $\begin{cases} x_{n-m+1} = g_1(x_1, x_2, \dots, x_{n-m}), \\ x_{n-m+2} = g_2(x_1, x_2, \dots, x_{n-m}), \\ \vdots \\ x_n = g_m(x_1, x_2, \dots, x_{n-m}) \end{cases}$ 满足 $x_j = g_j(x_1, x_2, \dots, x_{n-m})$ ($j = n-m+1, n-m+2, \dots, n$) 且有 $\frac{\partial g_k}{\partial x_1}(x_1, \dots, x_{n-m}) = 0, \frac{\partial g_k}{\partial x_2}(x_1, \dots, x_{n-m}) = 0, \dots, \frac{\partial g_k}{\partial x_{n-m}}(x_1, \dots, x_{n-m}) = 0$ 将隐函数组代入 $f(\mathbf{x}_0)$ 得 $f(x_1, \dots, x_{n-m}, g_1(x_1, \dots, x_{n-m}), g_2(x_1, \dots, x_{n-m}), \dots, g_m(x_1, \dots, x_{n-m}))$ 因此 \mathbf{x}_0 是条件极值点转化为 $(x_1^0, x_2^0, \dots, x_{n-m}^0)$ 为上述函数的通常极值点。

令 \mathbf{x}_0' 则对 $i = 1, 2, \dots, n-m$

$\frac{\partial f(\mathbf{x}_0)}{\partial x_i} + \frac{\partial f(\mathbf{x}_0)}{\partial x_{n-m+1}} \frac{\partial g_1}{\partial x_i}(x_1^0, \dots, x_{n-m}^0) + \cdots + \frac{\partial f(\mathbf{x}_0)}{\partial x_n} \frac{\partial g_m}{\partial x_i}(x_1^0, \dots, x_{n-m}^0) = 0$ 令

$\$\\mathbf{g}(\mathbf{x})=(g_1(\mathbf{x}), g_2(\mathbf{x}), \dots, g_m(\mathbf{x}))^T$ 其中
 $\mathbf{x}=(x_1, x_2, \dots, x_{n-m})$ 将上述 $n-m$ 个等式写成向量形式，
有 $\frac{\partial f(\mathbf{x}_0)}{\partial x_1}, \frac{\partial f(\mathbf{x}_0)}{\partial x_2}, \dots, \frac{\partial f(\mathbf{x}_0)}{\partial x_{n-m}}$ 的线性组合为零，即
$$\left(\begin{array}{c} \frac{\partial f(\mathbf{x}_0)}{\partial x_1} \\ \frac{\partial f(\mathbf{x}_0)}{\partial x_2} \\ \vdots \\ \frac{\partial f(\mathbf{x}_0)}{\partial x_{n-m}} \end{array} \right) \cdot \left(\begin{array}{c} \frac{\partial g_1(\mathbf{x}_0)}{\partial x_1} & \frac{\partial g_1(\mathbf{x}_0)}{\partial x_2} & \cdots & \frac{\partial g_1(\mathbf{x}_0)}{\partial x_{n-m}} \\ \frac{\partial g_2(\mathbf{x}_0)}{\partial x_1} & \frac{\partial g_2(\mathbf{x}_0)}{\partial x_2} & \cdots & \frac{\partial g_2(\mathbf{x}_0)}{\partial x_{n-m}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g_m(\mathbf{x}_0)}{\partial x_1} & \frac{\partial g_m(\mathbf{x}_0)}{\partial x_2} & \cdots & \frac{\partial g_m(\mathbf{x}_0)}{\partial x_{n-m}} \end{array} \right) = 0$$

由 $\frac{\partial g_i(\mathbf{x}_0)}{\partial x_1} = -\frac{\partial g_i(\mathbf{x}_0)}{\partial x_{n-m+1}}$ 可得
$$\left(\begin{array}{c} \frac{\partial f(\mathbf{x}_0)}{\partial x_1} \\ \frac{\partial f(\mathbf{x}_0)}{\partial x_2} \\ \vdots \\ \frac{\partial f(\mathbf{x}_0)}{\partial x_{n-m}} \end{array} \right) \cdot \left(\begin{array}{c} \frac{\partial g_1(\mathbf{x}_0)}{\partial x_1} & \frac{\partial g_1(\mathbf{x}_0)}{\partial x_2} & \cdots & \frac{\partial g_1(\mathbf{x}_0)}{\partial x_{n-m}} \\ \frac{\partial g_2(\mathbf{x}_0)}{\partial x_1} & \frac{\partial g_2(\mathbf{x}_0)}{\partial x_2} & \cdots & \frac{\partial g_2(\mathbf{x}_0)}{\partial x_{n-m}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g_m(\mathbf{x}_0)}{\partial x_1} & \frac{\partial g_m(\mathbf{x}_0)}{\partial x_2} & \cdots & \frac{\partial g_m(\mathbf{x}_0)}{\partial x_{n-m}} \end{array} \right) = 0$$

令 $A = \left(\begin{array}{cccc} \frac{\partial f(\mathbf{x}_0)}{\partial x_1} & \frac{\partial f(\mathbf{x}_0)}{\partial x_2} & \cdots & \frac{\partial f(\mathbf{x}_0)}{\partial x_{n-m}} \\ \frac{\partial g_1(\mathbf{x}_0)}{\partial x_1} & \frac{\partial g_1(\mathbf{x}_0)}{\partial x_2} & \cdots & \frac{\partial g_1(\mathbf{x}_0)}{\partial x_{n-m}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g_m(\mathbf{x}_0)}{\partial x_1} & \frac{\partial g_m(\mathbf{x}_0)}{\partial x_2} & \cdots & \frac{\partial g_m(\mathbf{x}_0)}{\partial x_{n-m}} \end{array} \right)$
 $B = \left(\begin{array}{c} \frac{\partial g_1(\mathbf{x}_0)}{\partial x_{n-m+1}} \\ \frac{\partial g_2(\mathbf{x}_0)}{\partial x_{n-m+1}} \\ \vdots \\ \frac{\partial g_m(\mathbf{x}_0)}{\partial x_{n-m+1}} \end{array} \right)$
$$A \cdot B = 0$$

注意到 A 是一个 $m \times m$ 矩阵，我们可以将其记为 A 。将 A 表示为
$$A = \lambda_1 \lambda_2 \cdots \lambda_m$$

代入之前的式子 得
$$\left(\begin{array}{c} \frac{\partial f(\mathbf{x}_0)}{\partial x_1} \\ \frac{\partial f(\mathbf{x}_0)}{\partial x_2} \\ \vdots \\ \frac{\partial f(\mathbf{x}_0)}{\partial x_{n-m}} \end{array} \right) \cdot \left(\begin{array}{c} \frac{\partial g_1(\mathbf{x}_0)}{\partial x_1} & \frac{\partial g_1(\mathbf{x}_0)}{\partial x_2} & \cdots & \frac{\partial g_1(\mathbf{x}_0)}{\partial x_{n-m}} \\ \frac{\partial g_2(\mathbf{x}_0)}{\partial x_1} & \frac{\partial g_2(\mathbf{x}_0)}{\partial x_2} & \cdots & \frac{\partial g_2(\mathbf{x}_0)}{\partial x_{n-m}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g_m(\mathbf{x}_0)}{\partial x_1} & \frac{\partial g_m(\mathbf{x}_0)}{\partial x_2} & \cdots & \frac{\partial g_m(\mathbf{x}_0)}{\partial x_{n-m}} \end{array} \right) = 0$$

另外我们可以将 λ 改写成 $\lambda_1 \lambda_2 \cdots \lambda_m$ ，即
$$\left(\begin{array}{c} \frac{\partial f(\mathbf{x}_0)}{\partial x_1} \\ \frac{\partial f(\mathbf{x}_0)}{\partial x_2} \\ \vdots \\ \frac{\partial f(\mathbf{x}_0)}{\partial x_{n-m}} \end{array} \right) \cdot \left(\begin{array}{c} \frac{\partial g_1(\mathbf{x}_0)}{\partial x_1} & \frac{\partial g_1(\mathbf{x}_0)}{\partial x_2} & \cdots & \frac{\partial g_1(\mathbf{x}_0)}{\partial x_{n-m}} \\ \frac{\partial g_2(\mathbf{x}_0)}{\partial x_1} & \frac{\partial g_2(\mathbf{x}_0)}{\partial x_2} & \cdots & \frac{\partial g_2(\mathbf{x}_0)}{\partial x_{n-m}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g_m(\mathbf{x}_0)}{\partial x_1} & \frac{\partial g_m(\mathbf{x}_0)}{\partial x_2} & \cdots & \frac{\partial g_m(\mathbf{x}_0)}{\partial x_{n-m}} \end{array} \right) = 0$$

$\frac{\partial \varphi_m(\mathbf{x}_0)}{\partial x_{n-m+2}} \dots & \dots \\ \frac{\partial \varphi_m(\mathbf{x}_0)}{\partial x_n} \end{array} \right) = 0 \quad \text{tag{5}}$ 将 $\left(4\right), \left(5\right)$ 写成分量形式再加上约束条件即可证明。

拉格朗日乘子法

构造函数 $F(x_1, \dots, x_n, \lambda_1, \dots, \lambda_m) = f(\mathbf{x}) + \sum_{j=1}^m \lambda_j \varphi_j(\mathbf{x})$ 则上述求条件极值点的必要条件形式转化为 F 的通常极值的必要条件 $\begin{cases} \frac{\partial F(\mathbf{x}_0)}{\partial x_i} = 0 \quad (i=1, 2, \dots, n) \\ \frac{\partial F(\mathbf{x}_0)}{\partial \lambda_j} = 0 \quad (j=1, 2, \dots, m) \end{cases}$ 此即拉格朗日乘子法

From:
<https://wiki.cvbbacm.com/> - CVBB ACM Team



Permanent link:
https://wiki.cvbbacm.com/doku.php?id=2020-2021:teams:farmer_john:2sozx:E6%95%B0%E5%AD%A6:%E7%9F%A5%E8%AF%86%E7%82%B9&rev=1591960318

Last update: 2020/06/12 19:11