

类欧几里得算法

算法思想

我们设 $f(a,b,c,n)=\sum_{i=0}^n \lfloor \frac{ai+b}{c} \rfloor$

其中 a,b,c,n 为常数，我们需要一个 $O(\log n)$ 的算法。

如果 $a \geq c$ 或者 $b \geq c$ 我们可以将 a,b 对 c 取模来化简问题：

$$f(a,b,c,n)=\sum_{i=0}^n \lfloor \frac{ai+b}{c} \rfloor$$

$$=\sum_{i=0}^n \lfloor \frac{(\lfloor \frac{a}{c} \rfloor c + a \bmod c) + (\lfloor \frac{b}{c} \rfloor c + b \bmod c)}{c} \rfloor$$

$$=\lfloor \frac{n(n+1)}{2} \rfloor \lfloor \frac{a}{c} \rfloor + (n+1) \lfloor \frac{b}{c} \rfloor + \sum_{i=0}^n \lfloor \frac{(a \bmod c) + (b \bmod c)}{c} \rfloor$$

$$=\lfloor \frac{n(n+1)}{2} \rfloor \lfloor \frac{a}{c} \rfloor + (n+1) \lfloor \frac{b}{c} \rfloor + f(a \bmod c, b \bmod c, n)$$

这样我们就将前两个参数控制到一定比第三个参数小的形式了。

$$\sum_{i=0}^n \lfloor \frac{ai+b}{c} \rfloor = \sum_{i=0}^n \sum_{j=0}^{\lfloor \frac{ai+b}{c} \rfloor - 1} 1$$

然后我们交换和号：

$$\sum_{j=0}^{\lfloor \frac{an+b}{c} \rfloor - 1} \sum_{i=0}^n [j < \lfloor \frac{ai+b}{c} \rfloor]$$

对于里面的式子，我们可以变换一下：

$$j < \lfloor \frac{ai+b}{c} \rfloor \iff j+1 \leq \lfloor \frac{ai+b}{c} \rfloor \iff j+c \leq ai+b \iff j+c-b-1 \leq ai \iff \lfloor \frac{j+c-b-1}{a} \rfloor \leq i$$

这样我们设 $m = \lfloor \frac{an+b}{c} \rfloor$

原式变为 $f(a,b,c,n) = \sum_{j=0}^{m-1} \sum_{i=0}^n [i \leq \lfloor \frac{j+c-b-1}{a} \rfloor] = \sum_{j=0}^{m-1} (n - \lfloor \frac{j+c-b-1}{a} \rfloor) = nm - f(c, c-b-1, a, m-1)$ 然后第一个参数又比第三个大了，就一直取模这样，类似于求最大公约数。

算法实现

```
#include <bits/stdc++.h>
using namespace std;
typedef long long ll;
```

```
const ll mod=998244353,base=233;

ll Hash(int a,int b,int c,int n) {
    return (((((ll)a*base%mod)+b)*base%mod+c)*base%mod+n)%mod;
}
unordered_map<ll,int> F;

ll f(int a,int b,int c,int n) {
    if(!a) return (ll)b/c*(n+1)%mod;
    ll tmp=Hash(a,b,c,n);
    if(F.find(tmp)!=F.end()) return F[tmp];
    if(a>=c||b>=c) return
F[tmp]=(((ll)n*(n+1)/2%mod*(a/c)%mod+((ll)n+1)*(b/c)%mod)%mod+f(a%c,b%c,c,n
))%mod;
    int m=((ll)a*n+b)/c;
    return F[tmp]=((ll)n*m%mod-f(c,c-b-1,a,m-1)+mod)%mod;
}
int n,a,b,c;
int main() {
    int t;
    scanf("%d",&t);
    while(t--) {
        scanf("%d %d %d %d",&n,&a,&b,&c);
        printf("%lld\n",f(a,b,c,n));
    }
    return 0;
}
```

代码练习

先设 $f(a,b,c,n)=\sum_{i=0}^n \lfloor \frac{ai+b}{c} \rfloor$

$\lfloor \frac{a}{c} \rfloor$

1.求 $g(a,b,c,n)=\sum_{i=0}^n \lfloor \frac{ai+b}{c} \rfloor^2$ 和

$h(a,b,c,n)=\sum_{i=0}^n i \lfloor \frac{ai+b}{c} \rfloor$

当 $a=0$ 时, $g(a,b,c,n)=(n+1)\lfloor \frac{b}{c} \rfloor^2 + \lfloor \frac{n(n+1)}{2} \rfloor \lfloor \frac{b}{c} \rfloor$

当 $a \geq c$ 或 $b \geq c$ 时, $g(a,b,c,n)=\sum_{i=0}^n \{ (\lfloor \frac{i(a \bmod c)+b \bmod c}{c} \rfloor + \lfloor \frac{a}{c} \rfloor + \lfloor \frac{b}{c} \rfloor)^2 \}$

$=\sum_{i=0}^n \{ (\lfloor \frac{i(a \bmod c)+b \bmod c}{c} \rfloor + \lfloor \frac{a}{c} \rfloor + \lfloor \frac{b}{c} \rfloor)^2 \} + 2(\lfloor \frac{a}{c} \rfloor + \lfloor \frac{b}{c} \rfloor) \sum_{i=0}^n \lfloor \frac{i(a \bmod c)+b \bmod c}{c} \rfloor + \sum_{i=0}^n \{ (\lfloor \frac{a}{c} \rfloor + \lfloor \frac{b}{c} \rfloor)^2 \}$

$=g(a \bmod c, b \bmod c, c, n) + 2 \lfloor \frac{a}{c} \rfloor h(a \bmod c, b \bmod c, c, n) + 2 \lfloor \frac{b}{c} \rfloor f(a \bmod c, b \bmod c, c, n) + \sum_{i=0}^n \{ (\lfloor \frac{a}{c} \rfloor + \lfloor \frac{b}{c} \rfloor)^2 \} + 2 \lfloor \frac{a}{c} \rfloor \lfloor \frac{b}{c} \rfloor \sum_{i=0}^n i + \sum_{i=0}^n \lfloor \frac{b}{c} \rfloor^2$

$$g(a \bmod c, b \bmod c, n) = 2 \lfloor \frac{a}{c} \rfloor h(a \bmod c, b \bmod c, n) + 2 \lfloor \frac{b}{c} \rfloor f(a \bmod c, b \bmod c, n) +$$

$$+ \lfloor \frac{n(n+1)(2n+1)}{6} \rfloor \lfloor \frac{a}{c} \rfloor^2 + n(n+1) \lfloor \frac{a}{c} \rfloor \lfloor \frac{b}{c} \rfloor + (n+1) \lfloor \frac{b}{c} \rfloor^2$$

$$h(a, b, c, n) = \sum_{i=0}^n i \lfloor \frac{i(a \bmod c) + b \bmod c}{c} \rfloor + i \lfloor \frac{a}{c} \rfloor \lfloor \frac{b}{c} \rfloor + \lfloor \frac{b}{c} \rfloor$$

$$= h(a \bmod c, b \bmod c, n) + \lfloor \frac{n(n+1)(2n+1)}{6} \rfloor \lfloor \frac{a}{c} \rfloor + \lfloor \frac{n(n+1)}{2} \rfloor \lfloor \frac{b}{c} \rfloor$$

当 $a \leq c$ 且 $b \leq c$ 时, 仍设 $m = \lfloor \frac{a+b}{c} \rfloor$

$$g(a, b, c, n) = 2 \sum_{i=0}^n \sum_{j=1}^{\lfloor \frac{ai+b}{c} \rfloor} j - \sum_{i=0}^n \lfloor \frac{ai+b}{c} \rfloor$$

$$= -f(a, b, c, n) + 2 \sum_{i=0}^n \sum_{j=1}^m j [j \leq \lfloor \frac{ai+b}{c} \rfloor]$$

$$= -f(a, b, c, n) + 2 \sum_{i=0}^n \sum_{j=1}^{m-1} (j+1) [(j+1)c \leq ai+b+1]$$

$$= -f(a, b, c, n) + 2 \sum_{j=0}^{m-1} (j+1) \sum_{i=0}^n [i > \lfloor \frac{jc+c-b-1}{a} \rfloor]$$

$$= -f(a, b, c, n) + 2 \sum_{j=0}^{m-1} (j+1) (n - \lfloor \frac{jc+c-b-1}{a} \rfloor)$$

$$= nm(m+1) - f(a, b, c, n) - 2h(c, -b-1, a, m)$$

$$h(a, b, c, n) = \sum_{i=0}^n i \sum_{j=1}^m j [j \leq \lfloor \frac{ai+b}{c} \rfloor]$$

$$= \sum_{i=0}^n i \sum_{j=0}^{m-1} [(j+1)c \leq ai+b+1]$$

$$= \sum_{j=0}^{m-1} \sum_{i=0}^n i [i \leq \lfloor \frac{jc+c-b-1}{a} \rfloor]$$

$$= \sum_{j=0}^{m-1} (\lfloor \frac{n(n+1)}{2} \rfloor - \sum_{i=0}^n i [i \leq \lfloor \frac{jc+c-b-1}{a} \rfloor])$$

$$= \sum_{j=0}^{m-1} (\lfloor \frac{n(n+1)}{2} \rfloor - \lfloor \frac{\lfloor \frac{jc+c-b-1}{a} \rfloor (\lfloor \frac{jc+c-b-1}{a} \rfloor + 1)}{2} \rfloor)$$

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