

类欧几里得算法

算法思想

我们设 $f(a,b,c,n)=\sum_{i=0}^n \lfloor \frac{ai+b}{c} \rfloor$

其中 a,b,c,n 为常数，我们需要一个 $O(\log n)$ 的算法。

如果 $a \geq c$ 或者 $b \geq c$ 我们可以将 a,b 对 c 取模来化简问题：

$$f(a,b,c,n)=\sum_{i=0}^n \lfloor \frac{ai+b}{c} \rfloor$$

$$=\sum_{i=0}^n \lfloor \frac{\lfloor \frac{a}{c} \rfloor c + a \bmod c}{c} \rfloor + \lfloor \frac{b}{c} \rfloor + \lfloor \frac{a \bmod c + b \bmod c}{c} \rfloor$$

$$=\lfloor \frac{n(n+1)}{2} \rfloor \lfloor \frac{a}{c} \rfloor + (n+1) \lfloor \frac{b}{c} \rfloor + \sum_{i=0}^n \lfloor \frac{(a \bmod c) + (b \bmod c)}{c} \rfloor$$

$$=\lfloor \frac{n(n+1)}{2} \rfloor \lfloor \frac{a}{c} \rfloor + (n+1) \lfloor \frac{b}{c} \rfloor + f(a \bmod c, b \bmod c, c, n)$$

这样我们就将前两个参数控制到一定比第三个参数小的形式了。

$$\sum_{i=0}^n \lfloor \frac{ai+b}{c} \rfloor = \sum_{i=0}^n \sum_{j=0}^{\lfloor \frac{ai+b}{c} \rfloor - 1} 1$$

然后我们交换和号：

$$\sum_{j=0}^{\lfloor \frac{an+b}{c} \rfloor - 1} \sum_{i=0}^n [j < \frac{ai+b}{c}]$$

对于里面的式子，我们可以变换一下：

$$j < \frac{ai+b}{c} \iff j+1 \leq \frac{ai+b}{c} \iff j+1 \leq \frac{ai+b}{c} \iff jc+c \leq ai+b \iff jc+c-b-1 \leq ai \iff \lfloor \frac{jc+c-b-1}{a} \rfloor \leq i$$

$$\text{所以我们设 } m = \lfloor \frac{an+b}{c} \rfloor$$

原式变为 $f(a,b,c,n)=\sum_{j=0}^{m-1} \sum_{i=0}^n [i \leq \lfloor \frac{jc+c-b-1}{a} \rfloor]$
 $=\sum_{j=0}^{m-1} (n - \lfloor \frac{jc+c-b-1}{a} \rfloor) = nm - f(c, c-b-1, a, m)$ 然后第一个参数又比第三个大了，就一直取模这样，类似于求最大公约数。

算法实现

```
#include <bits/stdc++.h>
using namespace std;
typedef long long ll;
```

```
const ll mod=998244353,base=233;

ll Hash(int a,int b,int c,int n) {
    return (((((ll)a*base%mod)+b)*base%mod+c)*base%mod+n)%mod;
}
unordered_map<ll,int> F;

ll f(int a,int b,int c,int n) {
    if(!a) return (ll)b/c*(n+1)%mod;
    ll tmp=Hash(a,b,c,n);
    if(F.find(tmp)!=F.end()) return F[tmp];
    if(a>=c||b>=c) return
F[tmp]=(((ll)n*(n+1)/2%mod*(a/c)%mod+((ll)n+1)*(b/c)%mod)%mod+f(a%c,b%c,c,n
))%mod;
    int m=((ll)a*n+b)/c;
    return F[tmp]=((ll)n*m%mod-f(c,c-b-1,a,m-1)+mod)%mod;
}
int n,a,b,c;
int main() {
    int t;
    scanf("%d",&t);
    while(t--) {
        scanf("%d %d %d %d",&n,&a,&b,&c);
        printf("%lld\n",f(a,b,c,n));
    }
    return 0;
}
```

代码练习

先设 $f(a,b,c,n)=\sum_{i=0}^n \lfloor \frac{ai+b}{c} \rfloor$

1.求 $g(a,b,c,n)=\sum_{i=0}^n \lfloor \frac{ai+b}{c} \rfloor^2$ 和
 $h(a,b,c,n)=\sum_{i=0}^n i \lfloor \frac{ai+b}{c} \rfloor$

当 $a=0$ 时, $g(a,b,c,n)=(n+1)\lfloor \frac{b}{c} \rfloor^2 + \lfloor \frac{n(n+1)}{2} \rfloor \lfloor \frac{b}{c} \rfloor$

当 $a \geq c$ 或 $b \geq c$ 时, $g(a,b,c,n)=\sum_{i=0}^n \lfloor \frac{i(a \bmod c)+b \bmod c}{c} \rfloor^2 + \lfloor \frac{a}{c} \rfloor \lfloor \frac{b}{c} \rfloor^2 + \lfloor \frac{a}{c} \rfloor \lfloor \frac{b}{c} \rfloor$

$=\sum_{i=0}^n \lfloor \frac{i(a \bmod c)+b \bmod c}{c} \rfloor^2 + 2\lfloor \frac{a}{c} \rfloor \lfloor \frac{b}{c} \rfloor + \lfloor \frac{a}{c} \rfloor \lfloor \frac{b}{c} \rfloor$

$=g(a \bmod c, b \bmod c, c, n) + 2\lfloor \frac{a}{c} \rfloor \lfloor \frac{b}{c} \rfloor + h(a \bmod c, b \bmod c, c, n) + 2\lfloor \frac{a}{c} \rfloor \lfloor \frac{b}{c} \rfloor$

$=g(a \bmod c, b \bmod c, c, n) + 2\lfloor \frac{a}{c} \rfloor \lfloor \frac{b}{c} \rfloor + h(a \bmod c, b \bmod c, c, n) + 2\lfloor \frac{a}{c} \rfloor \lfloor \frac{b}{c} \rfloor$

$$c\} \lfloor f(a \bmod c, b \bmod c, c, n) \rfloor$$

$$+ \frac{n(n+1)(2n+1)}{6} \lfloor \frac{a}{c} \rfloor^2 + n(n+1) \lfloor \frac{a}{c} \rfloor \lfloor \frac{b}{c} \rfloor + (n+1) \lfloor \frac{b}{c} \rfloor^2$$

$$h(a, b, c, n) = \sum_{i=0}^n i \lfloor \frac{i(a \bmod c) + b \bmod c}{c} \rfloor + i \lfloor \frac{a}{c} \rfloor \lfloor \frac{b}{c} \rfloor$$

$$= h(a \bmod c, b \bmod c, c, n) + \frac{n(n+1)(2n+1)}{6} \lfloor \frac{a}{c} \rfloor + \frac{n(n+1)}{2} \lfloor \frac{b}{c} \rfloor$$

当 $a \leq c$ 且 $b \leq c$ 时，仍设 $m = \lfloor \frac{a+b}{c} \rfloor$

$$g(a, b, c, n) = 2 \sum_{i=0}^n \sum_{j=1}^{\lfloor \frac{ai+b}{c} \rfloor} j - \sum_{i=0}^n \lfloor \frac{ai+b}{c} \rfloor$$

$$= -f(a, b, c, n) + 2 \sum_{i=0}^n \sum_{j=1}^m [j \leq \lfloor \frac{ai+b}{c} \rfloor]$$

$$= -f(a, b, c, n) + 2 \sum_{i=0}^n \sum_{j=1}^{m-1} (j+1) [(j+1)c \leq ai+b+1]$$

$$= -f(a, b, c, n) + 2 \sum_{j=0}^{m-1} (j+1) \sum_{i=0}^n [i > \lfloor \frac{jc+c-b-1}{a} \rfloor]$$

$$= -f(a, b, c, n) + 2 \sum_{j=0}^{m-1} (j+1) (n - \lfloor \frac{jc+c-b-1}{a} \rfloor)$$

$$= nm(m+1) - f(a, b, c, n) - 2h(c, -b-1, a, m)$$

$$h(a, b, c, n) = \sum_{i=0}^n i \sum_{j=1}^m [j \leq \lfloor \frac{ai+b}{c} \rfloor]$$

$$= \sum_{i=0}^n i \sum_{j=0}^{m-1} [(j+1)c \leq ai+b+1]$$

$$= \sum_{j=0}^{m-1} \sum_{i=0}^n [i \leq \lfloor \frac{jc+c-b-1}{a} \rfloor]$$

$$= \sum_{j=0}^{m-1} (\frac{n(n+1)}{2} - \sum_{i=0}^n [i \leq \lfloor \frac{jc+c-b-1}{a} \rfloor])$$

$$= \sum_{j=0}^{m-1} (\frac{n(n+1)}{2} - \lfloor \frac{\lfloor \frac{jc+c-b-1}{a} \rfloor (\lfloor \frac{jc+c-b-1}{a} \rfloor + 1)}{2} \rfloor)$$

$$= \frac{\sum_{j=0}^{m-1} n(n+1) - \sum_{j=0}^{m-1} \lfloor \frac{jc+c-b-1}{a} \rfloor (\lfloor \frac{jc+c-b-1}{a} \rfloor + 1)}{2}$$

$$= \frac{1}{2} [mn(n+1) - g(c, c-b-1, a, m-1) - f(c, c-b-1, a, m-1)]$$

注意负数那里会导致计算错误，比如 $\frac{-1}{2} = \frac{1}{2} = 0$

所以在实际计算中，要规避掉参数为负的情况，这个具体情况具体分析。

```
#include <bits/stdc++.h>
using namespace std;
typedef long long ll;

const ll mod=998244353, inv2=(mod+1)>>1, inv3=332748118;
```

```

struct Ans {
    ll f,s,t;
}ans,now,las;
Ans getans(int a,int b,int c,int n) {
    if(!a) {
        now.f=(ll)b/c*(n+1)%mod;
        now.s=(ll)(b/c)*(b/c)%mod*(n+1)%mod;
        now.t=((ll)n+1)*n/2%mod*(b/c)%mod;
    }
    else if(a>=c||b>=c) {
        las=getans(a%c,b%c,c,n);
        now.f=((ll)n*(n+1)/2%mod*(a/c)%mod+((ll)n+1)*(b/c)%mod)%mod+las.f)%mod;
        ll
        tmp1=((las.s+2ll*(a/c)%mod*las.t%mod)%mod+2ll*(b/c)%mod*las.f%mod)%mod;
        ll
        tmp2=((ll)n*(n+1)%mod*(2ll*n+1)%mod*inv2%mod*inv3%mod*(a/c)%mod*(a/c)%mod+
        (ll)n*(n+1)%mod*(a/c)%mod*(b/c)%mod)%mod+(ll)(n+1)*(b/c)%mod*(b/c)%mod)%mod
        ;
        now.s=(tmp1+tmp2)%mod;
        now.t=((ll)n*(n+1)/2%mod*(b/c)%mod+((ll)n+1)*n%mod*inv2%mod*((ll)n*2+1)%mo
        d*inv3%mod*(a/c)%mod)%mod+las.t)%mod;
    }else {
        ll m=((ll)a*n+b)/c;
        las=getans(c,c-b-1,a,m-1);
        now.f=((ll)n*m%mod-las.f+mod)%mod;
        now.s=((ll)n*m%mod*(m+1)%mod-
        now.f+mod-2ll*las.t%mod+mod-2ll*las.f%mod+mod)%mod;
        now.t=((ll)m*n%mod*(n+1)%mod-las.f+mod-las.s+mod)%mod*inv2%mod;
    }
    return now;
}
int n,a,b,c;
int main() {
    int t;
    scanf("%d",&t);
    while(t--) {
        scanf("%d %d %d %d",&n,&a,&b,&c);
        ans=getans(a,b,c,n);
        printf("%lld %lld %lld\n",ans.f,ans.s,ans.t);
        //printf("%lld %lld %lld\n",f(a,b,c,n),g(a,b,c,n),h(a,b,c,n));
    }
    return 0;
}
/*
10
7 7 7 7
8 0 10 4
8 2 4 3
3 4 5 3
0 3 10 1

```

1 0 0 7
3 9 10 1
8 5 5 4
5 4 0 9
10 4 4 9
*/

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